

## APPLICATION OF THE INTEGRAL II: FUTURE AND PRESENT VALUE OF A CONTINUOUS INCOME STREAM

Let us review some basic formulae from a few weeks ago involving the return on money deposited in a bank paying a given rate of interest. If an initial amount of  $M$  dollars is deposited in a bank paying an interest rate of  $r$  per year compounded continuously, the future value of this money is given by the formula

$$(0.1) \quad \text{Future value} = Me^{rt}.$$

Conversely, if one aims to obtain an amount of  $N$  dollars  $t$  years down the road from an account that accrues interest at the annual rate of  $r$ , then the present value for this amount, i.e., the amount of money that one needs to put in today is

$$(0.2) \quad \text{Present value} = Ne^{-rt}.$$

The calculation of future value above was made under the assumption that once the initial deposit is made, there is no future deposits or withdrawals. But a much more realistic framework is one where a sequence of future deposits is made into the account after the initial one and over a long period of time. If the deposits are made regularly enough and the time between deposits is relatively short compared to the overall lifetime of the account, we can think of the money as flowing continuously into the account rather than in a large number of discrete chunks. We will refer to this scenario as a continuous income stream. The goal of this note is to deduce a closed-form integral formula for future and present values of a continuous income stream.

### 1. FUTURE VALUE

Suppose that a person plans to retire in  $T$  years and arranges for money to be deposited continuously in the bank at the rate of  $\$S(t)$  in year  $t$ . If the interest compounded at the annual rate of  $r$ , what is the future value of the income in  $T$  years?

To deal with this problem, we choose a large integer  $n$  and partition the time interval  $[0, T]$  into  $n$  equal subintervals each of length  $\Delta t = T/n$ . Let

$$t_0 = 0, \quad t_1 = \frac{T}{n}, \quad t_2 = \frac{2T}{n}, \quad \dots, \quad t_n = \frac{nT}{n} = T$$

be the endpoints of the subintervals. Since the money is being deposited continuously, the amount that flows into the account during the time interval  $[0, t_1]$  is equal to  $S(0)\Delta t$ . This money remains in the account for  $T - t_1$  years and by the formula (0.1), its future value at the end of that time is

$$S(0) \cdot \Delta t \cdot e^{r(T-t_1)}.$$

Actually, the expression above is only an approximation of the future value because we have only computed it starting from time  $t_1$ , and ignored the interest accumulated during the time period  $[0, t_1]$ . However, if  $n$  is large, that period is very small relative to the total  $T$  years, and we get a good approximation.

Next, we consider the time interval  $[t_1, t_2]$ , which also has length  $\Delta t$ . The total amount deposited in the bank during that period is equal to  $S(t_1)\Delta t$ . It remains in the account for  $T - t_2$  years and its approximate future value is given by formula (0.1) as

$$S(t_1) \cdot \Delta t \cdot e^{r(T-t_2)}.$$

Continuing in this way, we can find the approximate future value of the amount deposited in each one of the  $n$  subintervals, the last one being given by

$$S(t_{n-1}) \cdot \Delta t \cdot e^{r(T-t_n)}.$$

If we add up the future values of the income in all the  $n$  subintervals, we obtain the amount

$$\begin{aligned} & S(0) \cdot \Delta t \cdot e^{r(T-t_1)} + S(t_1) \cdot \Delta t \cdot e^{r(T-t_2)} + \dots + S(t_{n-1}) \cdot \Delta t \cdot e^{r(T-t_n)} \\ &= \Delta t \left[ S(t_0)e^{r(T-t_0)+r(t_1-t_0)} + S(t_1)e^{r(T-t_1)+r(t_1-t_2)} + \dots + S(t_{n-1})e^{r(T-t_{n-1})+r(t_n-t_{n-1})} \right] \\ &= e^{-\Delta t} \left( \sum_{j=0}^{n-1} S(t_j)e^{r(T-t_j)} \Delta t \right). \end{aligned}$$

The first factor  $e^{\Delta t} = e^{T/n} \rightarrow 1$  as  $n \rightarrow \infty$ . The second factor is the left Riemann sum of the function  $S(t)e^{r(T-t)}$  from  $t = 0$  to  $t = T$ , which converges to the definite integral for this function as  $n \rightarrow \infty$ .

The discussion so far is summarized as follows: the future value of a continuous income stream flowing at the rate of  $S(t)$  dollars per year for  $T$  years, earning interest at an annual rate  $r$ , compounded continuously, is given by

$$(1.1) \quad \text{Future value of a continuous income stream} = \int_0^T S(t)e^{r(T-t)} dt.$$

## 2. PRESENT VALUE

Repeating a similar argument as above, we can compute the present value of a continuous income stream that will generate  $I(t)$  dollars in year  $t$ , with interest compounded continuously at an annual rate  $r$ . More specifically, if I open an account that gives interest compounded continuously at the rate  $r$ , with the objective of earning  $I(t)$  dollars in year  $t$  for every year until the year  $T$ , the money that I need to put in today is

$$(2.1) \quad \text{present value of a continuous income stream} = \int_0^T I(t)e^{-rt} dt.$$

## 3. EXAMPLES

**3.1. Example 1.** Suppose that a company manufactures a certain type of machine, and it estimates that each machine will generate a continuous income stream whose

rate in the  $t$ th year of operation will be  $15 - 2t$  million dollars per year. Assuming that the lifetime of a machine is about 7 years and that the money can be invested at the annual rate of 8%, compounded continuously, find the fair market price of each machine.

**Solution.** In this situation it is reasonable to assume that the fair market price of each machine is the present value of the continuous income stream with

$$S(t) = 15 - 2t, \quad r = 0.08, \quad T = 7.$$

Using formula (2.1) we find that this price is

$$\int_0^7 (15 - 2t)e^{-0.08t} dt = 15 \int_0^7 e^{-0.08t} dt - 2 \int_0^7 te^{-0.08t} dt.$$

The first integral is easy, the second one involves integration by parts. Compute these integrals, and check that the fair market price of each machine is about \$46.363 million.  $\square$

**3.2. Example 2.** If you plan to deposit \$1200 per year continuously into an account that pays 6% annual interest compounded continuously, what would be the amount in the account at the end of 10 years?

**Solution.** We need to calculate the future value of a continuous income stream, hence formula (1.1) gives

$$\text{future value} = \int_0^{10} 1200e^{0.06(10-t)} dt = 20,000(e^{0.6} - 1) \approx 16,422.$$

$\square$