Solutions to Math 105 Practice Midterm1, Spring 2011

1. Short answer questions

(1) Assume $G(t)$ is an antiderivative of $\sqrt{t^4+1}$, that is, $G'(t) = \sqrt{t^4+1}$. Then

$$\begin{aligned} f'(x) &= \tfrac{d}{dx}\int_x^{x^2} \sqrt{t^4+1}dt = \tfrac{d}{dx}[G(t)|_x^{x^2}] = \tfrac{d}{dx}[G(x^2) - G(x)] \\ &= G'(x^2)\cdot(2x) - G'(x) = \sqrt{(x^2)^4+1}\cdot(2x) - \sqrt{x^4+1} \\ &= \sqrt{x^8+1}\cdot(2x) - \sqrt{x^4+1}. \end{aligned}$$

Thus $f'(1) = \sqrt{1+1}\cdot 2 - \sqrt{1+1} = \sqrt{2}$.

(2). According to the problem, we have

$$\begin{aligned} \frac{1}{b-0}\int_0^b (3x^2-6x+2)dx &= \tfrac{1}{b}[x^3-3x^2+2x]|_0^b = \tfrac{1}{b}(b^3-3b^2+2b) \\ &= b^2-3b+2 = (b-2)(b-1) = 0. \end{aligned}$$

Thus $b = 1$ or $b = 2$.

(3). First, $\Delta x = \frac{b-a}{n} = \frac{4-1}{3} = 1$ and $x_0 = 1$, $x_1 = 2$, $x_2 = 3$, $x_3 = 4$. Then by Trapezoid Rule approximation formula we have

$$\begin{aligned} T_3 &= [\frac{1}{2}f(x_0) + f(x_1) + f(x_2) + \frac{1}{2}f(x_3)]\cdot\Delta x \\ &= [\frac{1}{2}\cos(\pi) + 2\cdot\cos(\frac{\pi}{2}) + 3\cdot\cos(\frac{\pi}{3}) + \frac{1}{2}\cdot 4\cdot\cos(\frac{\pi}{4})]\cdot 1 \\ &= (\frac{1}{2}\cdot(-1) + 2\cdot 0 + 3\cdot\frac{1}{2} + 2\cdot\frac{\sqrt{2}}{2}) = 1+\sqrt{2}. \end{aligned}$$

2. $\int_0^1 e^{(2x+e^x)}\,dx = \int_0^1 e^{2x}\cdot e^{e^x}\,dx$. Let $t = e^x$ and $dt = e^x dx$.When $x = 0$, $t = 1$; when $x = 1$, $t = e$. Then by substitution, we have

$$\int_0^1 e^{2x}\cdot e^{e^x}\,dx = \int_1^e te^t dt.$$

Now we need to use integration by parts. Let $u = t$, $du = dt$ and $dv = e^t dt$, $v = e^t$. Thus

$$\int_1^e te^t dt = te^t|_1^e - \int_1^e e^t dt = te^t|_1^e - e^t|_1^e = (e\cdot e^e - e) - (e^e - e) = (e-1)e^e.$$

3. The average value is $\frac{1}{\pi/2-0}\int_0^{\pi/2} |\sin\theta - \cos\theta| d\theta$. Notice that $\sin\theta \le \cos\theta$

on $[0,\, \frac{\pi}{4}]$ and $\sin\theta \geq \cos\theta$ on $[\frac{\pi}{4},\, \frac{\pi}{2}]$. Thus

$$\begin{aligned}
&\frac{1}{\pi/2-0}\int_0^{\pi/2} |\sin\theta - \cos\theta| d\theta \\
&\quad = \frac{2}{\pi}[\int_0^{\pi/4}(\cos\theta - \sin\theta)d\theta + \int_{\pi/4}^{\pi/2}(\sin\theta - \cos\theta)d\theta] \\
&\quad = \frac{2}{\pi}[(\sin\theta + \cos\theta)|_0^{\pi/4} + (-\cos\theta - \sin\theta)|_{\pi/4}^{\pi/2}] \\
&\quad = \frac{2}{\pi}[(\sin\frac{\pi}{4} + \cos\frac{\pi}{4} - \sin 0 - \cos 0) + ((-\cos\frac{\pi}{2} - \sin\frac{\pi}{2}) - (-\cos\frac{\pi}{4} - \sin\frac{\pi}{4}))] \\
&\quad = \frac{4(\sqrt{2}-1)}{\pi}.
\end{aligned}$$

4. (1). $\int x(\ln x)^2\, dx$. We are going to use integration by parts twice.

Let $u = (\ln x)^2$, $du = 2\ln x \cdot \frac{1}{x}dx$ and $dv = x \cdot dx$, $v = \frac{1}{2}x^2$. Thus

$$\int x(\ln x)^2\, dx = \frac{1}{2}x^2(\ln x)^2 - \int \frac{1}{2}x^2 \cdot 2 \cdot \ln x \cdot \frac{1}{x} = \frac{1}{2}x^2(\ln x)^2 - \int x\ln x dx.$$

Let $u = \ln x$, $du = \frac{1}{x}dx$ and $dv = x \cdot dx$, $v = \frac{1}{2}x^2$. Thus

$$= \frac{1}{2}x^2(\ln x)^2 - [\frac{1}{2}x^2\ln x - \int \frac{1}{2}x^2 \cdot \frac{1}{x}\, dx] = \frac{1}{2}x^2(\ln x)^2 - \frac{1}{2}x^2\ln x + \frac{1}{4}x^2 + C.$$

(2). $\int \frac{4x+4}{x(x+1)^2}\, dx = \int \frac{4(x+1)}{x(x+1)^2}\, dx = \int \frac{4}{x(x+1)}\, dx$. Assume

$$\frac{4}{x(x+1)} = \frac{A}{x} + \frac{B}{x+1} = \frac{A(x+1)+Bx}{x(x+1)} = \frac{(A+B)x+A}{x(x+1)}.$$

Compare the coefficients of term x and the constant term, we have $A+B = 0,\ A = 4$, that is $A = 4$, $B = -4$. Then

$$\int \frac{4}{x(x+1)}\, dx = 4\int \frac{1}{x}\, dx - 4\int \frac{1}{(x+1)}\, dx = 4\ln\frac{|x|}{|x+1|} + C.$$

5. Find a function $f(x)$ whose graph goes through the point $(0,\, 3)$ and whose slope at any point $(x,\, f(x))$ is

$$\lim_{n\to\infty}[(1 + (1+2\frac{x}{n})^3 + (1+2\frac{2x}{n})^3 + (1+2\frac{3x}{n})^3 + ... + (1+2\frac{(n-1)x}{n})^3)] \cdot \frac{x}{n}.$$

Step 1 We estimate the Riemann Sum given in the problem by definite integral. Assume the Riemann Sum is on the interval $[a,\, b]$, where $a = 0$, $b = x$, thus $\Delta x = \frac{b-a}{n} = \frac{x}{n}$. Then $x_k = a + k \cdot \Delta x = \frac{kx}{n}$ and $x_{k-1} = \frac{(k-1)x}{n}$. So the Riemann Sum can be rewritten as

$$\sum_{k=1}^{n}(1 + 2 \cdot \frac{(k-1)x}{n})^3 \cdot \frac{x}{n} = \sum_{k=1}^{n} g(x_{k-1}) \cdot \Delta x,$$

where $g(x) = (1+2x)^3$. So the Riemann Sum equal to the definite integral $\int_0^x g(t)\,dt = \int_0^x (1+2t)^3 dx$.

Step 2 The slope at point $(x,\, f(x))$ is $f'(x)$. Hence by step 1 we have $f'(x) = \int_0^x (1+2t)^3 dt$. Let $u = 2t+1$, $du = 2dt$ and $t = 0$, $u = 1$; $t = x$, $u = 1+2x$. Then

$$f'(x) = \int_0^x (1+2t)^3 dt = \int_1^{(1+2x)} u^3 \cdot \frac{1}{2} \cdot du = \frac{1}{8} u^4|_1^{1+2x} = \frac{1}{8}[(1+2x)^4 - 1].$$

Step 3 Then

$$f(x) = \int f'(x)\,dx = \int \frac{1}{8}[(1+2x)^4 - 1] dx = \frac{1}{80}(1+2x)^5 - \frac{1}{16}(1+2x) + C.$$

The last equality is from substitution. Since $f(0) = 3$, plugging in $x = 0$, $f(x) = 3$, we have $C = \frac{61}{20}$. Thus

$$f(x) = \frac{1}{80}(1+2x)^5 - \frac{1}{16}(1+2x) + \frac{61}{20}.$$