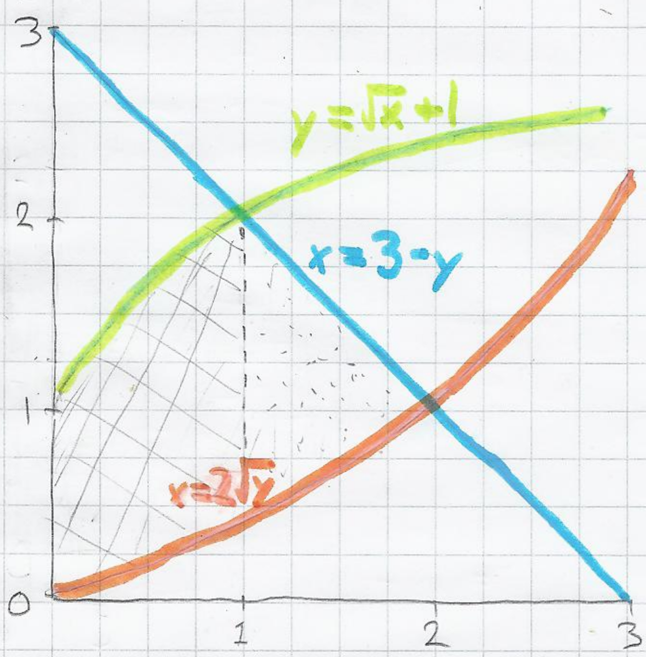
1. Find the area of the four-sided region in the first quadrant that is bounded on the left by the y-axis, below by the curve $x = 2\sqrt{y}$, above left by the curve $y = \sqrt{x} + 1$ and above right by the line $x = 3 - y$.

First we draw the graph:



The point where the curves $y = \sqrt{x} + 1$ and $x = 3 - y$ intersect is where $y = \sqrt{x} + 1 = 3 - x$.
If $\sqrt{x} + 1 = 3 - x$, then $\sqrt{x} = 2 - x$, and squaring both sides, $x = (2-x)^2 = 4 - 4x + x^2$, so $x^2 - 5x + 4 = 0$. This quadratic has roots at $x = 1$ and $x = 4$, and checking these in the original equation, $x = 1$ is the correct solution to $\sqrt{x} + 1 = 3 - x$, so $x = 1, y = 2$.
The curve $x = 2\sqrt{y}$ bounding the region from below intersects $x = 3 - y$ when $x = 2, y = 1$.

Now we can divide the region into two parts each of which is bounded between two curves:
The first is between the curves $y = x^2/4$ (rewriting $x = 2\sqrt{y}$) and $y = \sqrt{x} + 1$ with $0 \le x \le 1$
and the second is between the curves $y = x^2/4$ and $y = 3 - x$ (rewriting $x = 3 - y$) with $1 \le x \le 2$.

The first part is given by $\int_0^1 \left[(\sqrt{x}+1) - (x^2/4)\right]dx = \left(\frac{2}{3}x^{3/2} + x - \frac{x^3}{12}\right)\Big|_0^1 = 2/3 + 1 - 1/12 = 19/12$

The second part is given by $\int_1^2 \left[(3-x) - (x^2/4)\right]dx = \left(3x - \frac{x^2}{2} - \frac{x^3}{12}\right)\Big|_1^2 = (6 - 2 - 8/12) - (3 - 1/2 - 1/12) = 11/12$

So the total area of the region is $19/12 + 11/12 = 30/12 = 5/2$

2. Evaluate the following two integrals, or if they diverge, explain why:

(a) $\displaystyle\int_0^3 \frac{1}{(x-1)^{2/3}}\,dx$

The integrand is not defined at 1, so this is an improper integral. We split it into left and right parts:

$$\int_0^3 \frac{1}{(x-1)^{2/3}}\,dx = \lim_{a\to 1^-}\int_0^a \frac{1}{(x-1)^{2/3}}\,dx + \lim_{b\to 1^+}\int_b^3 \frac{1}{(x-1)^{2/3}}\,dx$$

Now we have $\displaystyle\int_0^a \frac{1}{(x-1)^{2/3}}\,dx \overset{\substack{u=x-1\\ du=dx}}{=} \int_{-1}^{a-1} \frac{1}{u^{2/3}}\,du = (3u^{1/3})\Big|_{-1}^{a-1} = (3(x-1)^{1/3})\Big|_0^a = 3(a-1)^{1/3} - 3,$

So $\displaystyle\lim_{a\to 1^-}\int_0^a \frac{1}{(x-1)^{2/3}}\,dx = \lim_{a\to 1^-}\left[3(a-1)^{1/3} - 3\right] = -3.$

For the other part, $\displaystyle\int_b^3 \frac{1}{(x-1)^{2/3}}\,dx = (3(x-1)^{1/3})\Big|_b^3 = 3\sqrt[3]{2} - 3(b-1)^{1/3}.$

So $\displaystyle\lim_{b\to 1^+}\int_b^3 \frac{1}{(x-1)^{2/3}}\,dx = \lim_{b\to 1^+}\left[3\sqrt[3]{2} - 3(b-1)^{1/3}\right] = 3\sqrt[3]{2}.$

Thus the improper integral is convergent and $\displaystyle\int_0^3 \frac{1}{(x-1)^{2/3}}\,dx = -3 + 3\sqrt[3]{2} = 3(\sqrt[3]{2} - 1)$

(b) $\displaystyle\int_{-\infty}^{\infty} \frac{x^2}{9+x^6}\,dx$

Pick a real number c and split the integral into the pieces left and right of c:

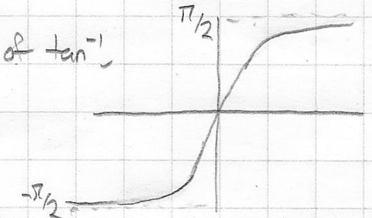
$$\int_{-\infty}^{\infty} \frac{x^3}{9+x^6}dx = \int_{-\infty}^{c} \frac{x^2}{9+x^6}dx + \int_{c}^{\infty} \frac{x^3}{9+x^6}dx = \lim_{a\to-\infty}\int_{a}^{c} \frac{x^2}{9+x^6}dx + \lim_{b\to\infty}\int_{c}^{b} \frac{x^3}{9+x^6}dx$$

For convenience, we choose $c=0$, so $\displaystyle\int_{-\infty}^{\infty} \frac{x^2}{9+x^6}dx = \lim_{a\to-\infty}\int_{a}^{0} \frac{x^3}{9+x^6}dx + \lim_{b\to\infty}\int_{0}^{b} \frac{x^2}{9+x^6}dx.$

For the left piece, $\displaystyle\int_{a}^{0} \frac{x^3}{9+x^6}dx \overset{\boxed{u=x^3,\ du=3x^2}}{=} \frac{1}{3}\int_{a^3}^{0} \frac{1}{9+u^2}du = \frac{1}{3}\left(\frac{1}{3}\tan^{-1}\left(\frac{u}{3}\right)\right)\Bigg|_{a^3}^{0} = \frac{1}{3}\left(\frac{1}{3}\tan^{-1}\left(\frac{x^3}{3}\right)\right)\Bigg|_{a}^{0}$

$$= -1/9 \tan^{-1}\left(\frac{a^3}{3}\right)$$

Looking at the graph of $\tan^{-1}$,



, $\displaystyle\lim_{x\to-\infty} \tan^{-1}(x) = -\pi/2,$

So we have $\displaystyle\lim_{a\to-\infty}\int_{a}^{0} \frac{x^2}{9+x^6}dx = \lim_{a\to-\infty}\left(-1/9 \tan^{-1}\left(a^3/3\right)\right) = \pi/18$

Noticing that the integrand is symmetric about 0 (an even function), we also have $\displaystyle\lim_{b\to\infty}\int_{0}^{b} \frac{x^2}{9+x^6} = \frac{\pi}{18}$

So the integral is convergent and $\displaystyle\int_{-\infty}^{\infty} \frac{x^2}{9+x^6}dx = \frac{\pi}{9}.$

3. We assume that the currency coming into the bank each day is mixed well so that the proportion of new currency that comes into the bank is the same as the proportion of new currency in overall circulation. The proportion of new bills in circulation at time t is $\frac{x(t)}{10^{10}}$.

Each day, $\$5\cdot10^7$ comes into the bank, $\frac{x(t)}{10^{10}}\cdot(5\cdot10^7) = \frac{5\cdot x(t)}{1000}$ of which is the new currency.
The bank replaces the old bills, so the amount of new bills introduced on day t is $\$5\cdot10^7 - \frac{5\cdot x(t)}{1000}$.
This is the rate of change of x, so we get the formula $x'(t) = \$5\cdot10^7 - \frac{1}{200}x(t)$.

This gives us the initial value problem $x'(t) = \$5\cdot10^7 - \frac{1}{200}x(t)$, $x(0)=0$.

The solution to the linear first order equation $x'(t) = \$5\cdot10^7 - \frac{1}{200}x(t)$ is $x(t) = \$\left(Ce^{\frac{-t}{200}} - \frac{5\cdot10^7}{(-1/200)}\right)$,
so $x(t) = \$(Ce^{\frac{-t}{200}} + 10^{10})$, where C is an arbitrary constant.
Now we find the constant C that corresponds to the initial value $x(0)=0$.
$x(0) = C\cdot e^{0/200} + 10^{10} = C + 10^{10} = 0$, so $C = -10^{10}$

Thus the amount of new bills at time t is given by $x(t) = \$(10^{10} - 10^{10}e^{-t/200}) = 10^{10}(1-e^{-t/200})$

To find the time when the proportion of new bills in circulation is 90%, we need to solve

$$.9 = \frac{x(t)}{10^{10}} = \frac{10^{10}(1-e^{-t/200})}{10^{10}} = 1-e^{-t/200}$$

This becomes $e^{-t/200} = 1-.9 = .1$.
Taking logarithms, $\ln(e^{-t/200}) = \frac{-t}{200} = \ln(.1)$, so $t = -200\ln(1/10) \approx 460.5$ days.