Math 105 Practice Exam 3 Solutions

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1. (a) *Evaluate* $\lim_{(x,y)\to(1,-1)} \frac{\sin(x^2+y)}{x^2+y}$ *or show that it doesn't exist.*
   We can use the substitution $u = x^2 + y$, so $u \to 1^2 - 1 = 0$:

$$\lim_{(x,y)\to(1,-1)} \frac{\sin(x^2+y)}{x^2+y} = \lim_{u\to 0} \frac{\sin(u)}{u} = \lim_{u\to 0} \frac{\sin(u)-\sin(0)}{u-0} = \frac{d}{dt}\sin(t)\Big|_{t=0} = \cos(0) = \boxed{1}.$$

   Here we used the definition of derivative, $f'(a) = \lim_{x\to a} \frac{f(x)-f(a)}{x-a}$, in reverse.

   (b) *Consider the area function* $A(x) = \int_1^x f(t)dt$*, with* $A(2) = 6$ *and* $A(3) = 5$*. Compute* $\int_3^2 f(t)dt$*.*
   By the Fundamental Theorem of Calculus, $\int_3^2 f(t)dt = A(2) - A(3) = 6 - 5 = \boxed{1}$.

   (c) *A self-employed software engineer estimates that her annual income over the next 10 years will steadily increase according to the formula* $70,000e^{0.1t}$*, where* t *is the time in years. She decides to save 10% of her income in an account paying 6% annual interest, compounded continuously. Treating the savings as a continuous income stream over a 10-year period, find the present value.*

$$PV = \int_0^{10} 7000e^{0.1t}e^{-0.06t}dt = 7000\int_0^{10} e^{0.04t}dt = 7000\cdot\frac{1}{0.04}e^{0.04t}\Big|_0^{10} = \boxed{175000(e^{0.4}-1)} (\approx 86,000)$$

   (d) *Draw the level curves of the graph of* $f(x,y) = 2x^2 + y^2$ *at the heights* $0, 1, 2$*.*
   For 1 and 2 it's an ellipse, for 0 it's just the point $(0,0)$.

   (e) *Evaluate* $\int_0^1 \frac{\cos(\sqrt{x})}{\sqrt{x}}dx$*.*

$$\int \frac{\cos(\sqrt{x})}{\sqrt{x}}dx \overset{u=\sqrt{x}}{=} 2\int \cos(u)du = 2\sin(u) + C = 2\sin(\sqrt{x}) + C,$$

$$\int_0^1 \frac{\cos(\sqrt{x})}{\sqrt{x}}dx = 2\lim_{c\to 0^+} \sin(\sqrt{x})\Big|_c^1 = 2\lim_{c\to 0^+}(\sin(1)-\sin(c)) = 2(\sin(1)-\sin(0)) = \boxed{2\sin(1)}$$

   (f) *Let* $f(x,y) = \frac{x+y}{x-y}$*. Use linear approximation to estimate* $f(2.95, 2.05)$*.*

$$f_x = \frac{-2y}{(x-y)^2}, \qquad f_y = \frac{2x}{(x-y)^2}$$

$$dz = f_x(3,2)dx + f_y(3,2)dy = (-4)\cdot(-0.05) + 6\cdot 0.05 = 0.5$$

$$\Rightarrow \quad f(2.95, 2.05) \approx f(3,2) + dz = 5 + 0.5 = \boxed{5.5}$$

2. *Evaluate* $\int \frac{x+2}{x(x^2-1)}dx$*.*

$$= -2\int \frac{1}{x}dx + \frac{3}{2}\int \frac{1}{x-1}dx + \frac{1}{2}\int \frac{1}{x+1}dx = \boxed{-2\ln|x| + \frac{3}{2}\ln|x-1| + \frac{1}{2}\ln|x+1| + C}$$

3. *Find the area of the region in the first quadrant bounded by* $y = \frac{1}{x}$, $y = 4x$, *and* $y = \frac{1}{2}x$.

$$= \int_0^{1/2} (4x - \frac{1}{2}x)dx + \int_{1/2}^{\sqrt{2}} (x^{-1} - \frac{1}{2}x)dx = \frac{7}{4}x^2\Big|_0^{1/2} + (\ln|x| - \frac{1}{4}x^2)\Big|_{1/2}^{\sqrt{2}}$$

$$= \frac{7}{16} + \ln(\sqrt{2}) - \frac{1}{2} - \ln(1/2) + \frac{1}{16} = \boxed{\frac{3}{2}\ln(2)}$$

4. *Find* k *such that* $f(x) = \frac{k}{(x+1)^3}$ *is a probability density function on the interval* $[0, \infty)$. *Then compute the probability that* $1 \le x \le 4$.

$$1 = \int_0^{\infty} \frac{k}{(x+1)^3}dx = k \lim_{c\to\infty} -\frac{1}{2}(x+1)^{-2}\Big|_0^c = \frac{k}{2} \lim_{c\to\infty} \left(1 - \frac{1}{(c+1)^2}\right) = \frac{k}{2} \Rightarrow \boxed{k = 2}$$

$$\Pr(1 \le X \le 4) = \int_1^4 \frac{2}{(x+1)^3}dx = -(x+1)^{-2}\Big|_1^4 = \frac{1}{4} - \frac{1}{25} = \frac{21}{100}$$

5. *Mothballs tend to evaporate at a rate proportional to their surface area. If* V *is the volume of a mothball, then its surface area is roughly* $V^{2/3}$. *Suppose that the mothball's volume* $V(t)$ *(as a function of times* t *in weeks) decreases at a rate that is twice its surface area, and that it initially has a volume of* 27 *cubic centimeters.*
Construct and solve an initial value problem for the volume $V(t)$. *Then determine if and when the mothball vanishes.*

$$\boxed{\frac{dV}{dt}} = -2V^{2/3} \Rightarrow \int V^{-2/3}dV = \int -2dt \Rightarrow 3V^{1/3} = -2t + C_1 \Rightarrow V = \left(-\frac{2}{3}t + C_2\right)^3$$

$$27 = V(0) = C_2^3 \Rightarrow C_2 = 3 \Rightarrow \boxed{V(t) = \left(3 - \frac{2}{3}t\right)^3}, \qquad V(t) = 0 \Rightarrow \boxed{t = \frac{9}{2}}$$

6. *Consider the surface* $z = f(x, y) = 1 + \frac{1}{\sqrt{xy}}$. *At the point on the surface above the point* $(x, y) = (4, 1)$, *what is the direction of steepest descent? Describe this direction with a unit vector in the xy-plane.*

$$\nabla f = \left\langle -\frac{1}{2x^{3/2}y^{1/2}}, -\frac{1}{2x^{1/2}y^{3/2}} \right\rangle \Rightarrow \nabla f = \left\langle -\frac{1}{16}, -\frac{1}{4} \right\rangle$$

$$\text{length} = \sqrt{\left(-\frac{1}{16}\right)^2 + \left(-\frac{1}{4}\right)^2} = \sqrt{\frac{4^2+1}{16^2}} = \frac{\sqrt{17}}{16}$$

$$\Rightarrow \text{unit vector of steepest descent} = -\frac{1}{\sqrt{17}/16}\left\langle -\frac{1}{16}, -\frac{1}{4} \right\rangle = \boxed{\left\langle \frac{1}{\sqrt{17}}, \frac{4}{\sqrt{17}} \right\rangle}$$

7. *By employing x semi-skilled workers and y skilled workers, a factory can assemble $\sqrt{4xy+y^2}$ custom-built computers per hour. The factory pays each semi-skilled worker \$8 per hour, and each skilled worker \$20 per hour. Determine the maximum number of computers the factory can assemble in an hour for a total labour cost of \$720.*

$$\text{optimize } f(x,y)=\sqrt{4xy+y^2} \text{ subject to } g(x,y)=8x+20y-720=0$$

$$\Rightarrow \nabla f=\left\langle \frac{2y}{\sqrt{4xy+y^2}}, \frac{2x+y}{\sqrt{4xy+y^2}} \right\rangle, \qquad \nabla g=\langle 8,20\rangle$$

$$\Rightarrow \text{ equations to solve: } \quad \frac{2y}{\sqrt{4xy+y^2}}=8\lambda, \quad \frac{2x+y}{\sqrt{4xy+y^2}}=20\lambda, \quad 8x+20y=720$$

$$\Rightarrow \text{ eliminate } \lambda\text{: } \lambda=\frac{y}{4\sqrt{4xy+y^2}}=\frac{2x+y}{20\sqrt{4xy+y^2}} \Rightarrow \frac{y}{4}=\frac{2x+y}{20} \Rightarrow y=\frac{1}{2}x$$

$$\Rightarrow \text{ plug into constraint: } 720=8x+20\cdot\frac{1}{2}x=18x \Rightarrow x=40 \Rightarrow y=20$$

To see if this is a max or a min, we take an arbitrary other point satisfying the constraint, like $(0,36)$, and compare the values: $f(0,36)=36$ while $f(40,20)=\sqrt{3200+400}=60$, so the maximum number of computers is $\boxed{60}$.

Note: Since the question did not specifically ask for Lagrange Multipliers, this question could have been done with one-variable calculus.

A trick to make things easier is to optimize f^2 instead of f, which will give the same x and y.

8. *Find and classify the critical points of $f(x,y)=7x^2-5xy+y^2+x-y+6$.*

$$f_x=14x-5y+1=0, \qquad f_y=-5x+2y-1=0$$

$$\Rightarrow y=\frac{14x+1}{5} \Rightarrow 0=-5x+2\cdot\frac{14x+1}{5}-1=-5x+\frac{28}{5}x+\frac{2}{5}-1=\frac{3}{5}x-\frac{3}{5}$$

$$\Rightarrow x=1, \quad y=3 \Rightarrow \boxed{(1,3) \text{ is the only critical point}}$$

$$f_{xx}=14, \quad f_{yy}=2, \quad f_{xy}=-5 \Rightarrow D(x,y)=28-(-5)^2=3$$

$$D(1,3)=3>0, \ f_{xx}(1,3)=14>0 \Rightarrow \boxed{(1,3) \text{ is a minimum}}$$

9. *Given the supply and demand curves*

$$p=D(q)=8-q, \qquad p=S(q)=\sqrt{q+1}+3,$$

find the equilibrium point and the consumer/producer surplus.

$$\sqrt{q+1}+3=8-q \Rightarrow q+1=(5-q)^2=25-10q+q^2 \Rightarrow 0=q^2-11q+24=(q-3)(q-8)$$

$$\Rightarrow \boxed{q_e=3, p_e=5}$$

Note that $q=8$ is not a solution, since $\sqrt{8+1}+3\neq 8-8$. Look at the graph: $q=8$ is where the line intersects the graph of $-\sqrt{q+1}+3$, which shows up because we squared both sides of the equation while solving.

$$CS=\int_0^3 (8-q)dq-3\cdot 5=24-\frac{9}{2}-15=\boxed{\frac{9}{2}}$$

$$PS=3\cdot 5-\int_0^3(\sqrt{q+1}+3)dq=15-\left(\frac{2}{3}(q+1)^{3/2}+3q\right)\Big|_0^3=15-\frac{16}{3}-9+\frac{2}{3}=\boxed{\frac{4}{3}}$$