Solutions to practice final1

1(a). **Answer**: Apply two path test. Let $y = mx^2$, $m \neq 1$ and $m \geq 0$. Then $\lim_{(x,\,y)\to(0,\,0)} \frac{xy^{1/2}}{x^2-y} = \lim_{(x,\,y)\to(0,\,0)} \frac{x\cdot\sqrt{m}x}{x^2-mx^2} = \frac{\sqrt{m}}{1-m}$. If $m = 4$, $Limit = -\frac{2}{3}$. If $m = 9$, $Limit = -\frac{3}{8}$. So by two path test, $\boxed{\text{the limit does not exist}}$.

1(b). **Answer**: $F'(x) = f(e^{x^2})(e^{x^2})' = f(e^{x^2}) \cdot e^{x^2} \cdot 2x$. Then $F'(1) = f(e^1) \cdot e \cdot 2 = 4 \cdot e \cdot 2 = \boxed{8e}$.

1(c). **Answer**: (A) describes an elliptic paraboloid. For 0, the equation of level curve is $\frac{x^2}{4} + \frac{y^2}{9} = 0$, it is just the point $(0,\,0)$. For 1, the equation of level curve is $\frac{x^2}{4} + \frac{y^2}{9} = 1$, it is a ellipse. There is no level curve corresponding -1.

1(d). **Answer**: Let $u = x^2$, then $du = 2x\,dx$. Then $\int xe^{-x^2}\,dx = \int \frac{1}{2}e^{-u}\,du = -\frac{1}{2}e^{-u}+C = -\frac{1}{2}e^{-x^2}+C$. Thus $\int_0^\infty xe^{-x^2}\,dx = \lim_{b\to\infty}\int_0^b xe^{-x^2}\,dx = \lim_{b\to\infty} -\frac{1}{2}e^{-x^2}|_0^b = \lim_{b\to\infty} -\frac{1}{2}[e^{-b^2} - e^0] = \frac{1}{2}$.

1(e). **Answer**: The distance between point $(x,\,y)$ and origin point is $f(x,\,y) = \sqrt{x^2+y^2}$. Let $(a,\,b) = (3,\,4)$, then $f(a,\,b) = 5$. We also have $\frac{\partial f}{\partial x} = \frac{x}{\sqrt{x^2+y^2}}$ and $\frac{\partial f}{\partial y} = \frac{y}{\sqrt{x^2+y^2}}$. Then $\frac{\partial f}{\partial x}(3,\,4) = \frac{3}{5}$ and $\frac{\partial f}{\partial y}(3,\,4) = \frac{4}{5}$. Thus $\sqrt{2.98^2+4.01^2} = f(2.98,\,4.01) \approx f(3,\,4) + \frac{\partial f}{\partial x}(3,\,4)\Delta x + \frac{\partial f}{\partial x}(3,\,4)\Delta y = 5 + 0.6\cdot(-0.02) + 0.8\cdot 0.01 = 5 - 0.012 + 0.008 = \boxed{4.996}$.

2(a). **Answer:** Use integration by parts. Let $u = \sin x$, $du = \cos x$ and $dv = e^x dx$, $v = e^x$. Then $\int e^x \sin x dx = e^x \sin x - \int e^x \cos x dx$. Use integration by parts again. Let $u = \cos x$, $du = -\sin x dx$ and $dv = e^x dx$, $v = e^x$. Then $\int e^x \sin x dx = e^x \sin x - e^x \cos x - \int e^x \sin x dx + C$. So $\int e^x \sin x dx = \frac{1}{2}(e^x \sin x - e^x \cos x) + C$. Thus $\int_0^{\pi/2} e^x \sin x\,dx = \frac{1}{2}(e^x \sin x - e^x \cos x)|_0^{\pi/2} = \frac{1}{2}[e^{\pi/2}\sin\frac{\pi}{2} - e^{\pi/2}\cos\frac{\pi}{2} - e^0 \sin 0 + e^0 \cos 0] = \boxed{\frac{1}{2}(e^{\frac{\pi}{2}} + 1)}$.

Answer: Divide the interval into two subintervals. Then $\Delta x = \frac{\pi}{4}$ and we have endpoints $0,\,\frac{\pi}{4},\,\frac{\pi}{2}$. Then by trapezoid rule we have $T_2 = \frac{1}{2}(e^0 \sin 0 + 2e^{\pi/4}\sin\pi/4 + e^{\pi/2}\sin\pi/2)\frac{\pi}{4} = \boxed{\frac{\pi}{8}[\sqrt{2}e^{\pi/4} + e^{\pi/2}]}$.

3. **Answer**: To find the intersection point of these two parabola, set $x^2 = x^2 - 4x + 4$, then $x = 1$. On $[0,\,1]$ $g(x) \geq f(x)$ and on $[1,\,3]$ $f(x) \geq g(x)$. Then $Area = \int_0^1 (x^2 - 4x + 4 - x^2)dx + \int_1^3 (x^2 - x^2 + 4x - 4)dx = \int_0^1 (-4x+4)dx + \int_1^3 (4x-4)dx = (-2x^2+4x)|_0^1 + (2x^2-4)|_1^3 = [-2+4] + [2\cdot 3^2 - 4\cdot 3 - 2\cdot 1^1 + 4\cdot 1] = \boxed{10}$.

4.**Answer**: (a). $\boxed{y'(t) = 0.12y(t) - A,\ y(0) = 50,000}$.

(b). The number of deers remains constant implies $0.12 \cdot 50,000 - A = 0$. So $\boxed{A = 6000}$.

(c). Suppose after T years the dears are extinct. Then

$$\int_0^T 7200e^{-0.12t}dt = 50,000.$$

Ont the other hand,

$$\begin{aligned}\int_0^T 7200e^{-0.12t}dt &= 7200\int_0^T e^{-0.12t}dt = 7200\frac{1}{-0.12}e^{-0.12t}|_0^T \\ &= 7200\frac{e^{-0.12T}-1}{-0.12} = 60000[1-e^{-0.12T}].\end{aligned}$$

Thus $60000[1-e^{-0.12T}] = 50000$, then $T = \frac{\ln 6}{0.12} \approx 15$. So $\boxed{\text{After 15 years the dears will go extinct}}$.

5. **Answer**: The profit function is

$$P(x,\, y) = x(300-x) + y(61 - \frac{3y}{x}) - 200x - y.$$

Now we are trying to calculate the critical points. We have

$$\frac{\partial P}{\partial x} = 100 - 2x + \frac{3y^2}{x^2} = 0 \tag{1}$$

$$\frac{\partial P}{\partial y} = 60 - \frac{6y}{x} = 0. \tag{2}$$

From (2), we have $\frac{y}{x} = 10$. Substituting into (1), we have $100 - 2x - 3(10)^2$. Thus $\boxed{x = 200,\ y = 2000}$.

6. **Answer**: The constraint in this case is $480x + 40y = 5000$. So we can apply the Lagrange multiplier method

$$F(x,\, y,\, \lambda) = 1000\sqrt{6x^2+y^2} + \lambda(480x + 40y - 5000).$$

Then

$$\frac{\partial F}{\partial x} = 1000\frac{6x}{\sqrt{6x^2+y^2}} + 480\lambda = 0, \tag{3}$$

$$\frac{\partial F}{\partial y} = 1000\frac{y}{\sqrt{6x^2+y^2}} + 40\lambda = 0 \tag{4}$$

and

$$\frac{\partial F}{\partial \lambda} = 480x + 4y - 5000 = 0. \tag{5}$$

From (3) and (4), we have

$$1000\frac{6x}{\sqrt{6x^2+y^2}} = -480\lambda \tag{6}$$

$$1000\frac{y}{\sqrt{6x^2+y^2}} = -40\lambda \tag{7}$$

Let (6)/(7), we have $x = 2y$. Substitute them into (5). Then we have $\boxed{x = 10,\ y = 5}$.

7. **Answer**: Set $0.05(q-10)^2 = 0.15q+2$. We have $q_e = 3$. The $p_e = 2.45$. So the equilibrium point is $\boxed{(3,\ 2.45)}$. Then the consumer surplus is

$$\int_0^3 0.05(q-10)^2 dq - p_e q_e = \frac{0.05}{3}(q-10)^3|_0^3 - 2.45 \times 3 = \boxed{3.6}.$$

8. **Answer**: Since $\frac{\partial f}{\partial x} = \frac{e^x}{1+e^y}$ and $\frac{\partial f}{\partial y} = \frac{-e^{x+y}}{(1+e^y)^2}$. Then the gradient at (0, 0) is $\nabla f(0,\ 0) =< \frac{1}{2}, -\frac{1}{4} >$. So the maximal increasing rate is $|\nabla f(0,\ 0)| = | < \frac{1}{2}, -\frac{1}{4} > | = \boxed{\frac{\sqrt{5}}{4}}$, and the corresponding direction is $\frac{1}{\frac{\sqrt{5}}{4}} < \frac{1}{2}, -\frac{1}{4} > = \boxed{< \frac{2}{\sqrt{5}}, -\frac{1}{\sqrt{5}} >}$.

9. **Answer**:

$$\begin{aligned}\int_0^\pi K|\cos(x)|dx &= K[\int_0^{\pi/2} \cos x dx + \int_{\pi/2}^{\pi} (-\cos x) dx] \\ &= K[\sin x|_0^{\pi/2} - \sin x|_{\pi/2}^{\pi}] = 2K = 1.\end{aligned}$$

So $\boxed{K = \frac{1}{2}}$.