18. $\lim \dfrac{n^2 - 2\sqrt{n} + 1}{1 - n - 3n^2} = \lim \dfrac{1 - \dfrac{2}{n\sqrt{n}} + \dfrac{1}{n^2}}{\dfrac{1}{n^2} - \dfrac{1}{n} - 3} = -\dfrac{1}{3}.$

24. $\lim\left(n - \sqrt{n^2 - 4n}\right) = \lim \dfrac{n^2 - (n^2 - 4n)}{n + \sqrt{n^2 - 4n}}$

$$= \lim \frac{4n}{n + \sqrt{n^2 - 4n}} = \lim \frac{4}{1 + \sqrt{1 - \dfrac{4}{n}}} = 2.$$

27. $a_n = \dfrac{(n!)^2}{(2n)!} = \dfrac{(1 \cdot 2 \cdot 3 \cdots n)(1 \cdot 2 \cdot 3 \cdots n)}{1 \cdot 2 \cdot 3 \cdots n \cdot (n+1) \cdot (n+2) \cdots 2n}$

$$= \frac{1}{n+1} \cdot \frac{2}{n+2} \cdot \frac{3}{n+3} \cdots \frac{n}{n+n} \leq \left(\frac{1}{2}\right)^n .$$

Thus $\lim a_n = 0$.

31. Let $a_1 = 3$ and $a_{n+1} = \sqrt{15 + 2a_n}$ for $n = 1, 2, 3, \ldots$. Then we have $a_2 = \sqrt{21} > 3 = a_1$. If $a_{k+1} > a_k$ for some k, then

$$a_{k+2} = \sqrt{15 + 2a_{k+1}} > \sqrt{15 + 2a_k} = a_{k+1}.$$

Thus, $\{a_n\}$ is increasing by induction. Observe that
$a_1 < 5$ and $a_2 < 5$. If $a_k < 5$ then

$$a_{k+1} = \sqrt{15 + 2a_k} < \sqrt{15 + 2(5)} = \sqrt{25} = 5.$$

Therefore, $a_n < 5$ for all n, by induction. Since $\{a_n\}$ is increasing and bounded above, it converges. Let $\lim a_n = a$. Then

$$a = \sqrt{15 + 2a} \Rightarrow a^2 - 2a - 15 = 0 \Rightarrow a = -3, \text{ or } a = 5.$$

Since $a > a_1$, we must have $\lim a_n = 5$.

12. Let

$$\sum_{n=1}^{\infty} \frac{1}{(2n-1)(2n+1)} = \frac{1}{1\times 3} + \frac{1}{3\times 5} + \frac{1}{5\times 7} + \cdots.$$

Since $\dfrac{1}{(2n-1)(2n+1)} = \dfrac{1}{2}\left(\dfrac{1}{2n-1} - \dfrac{1}{2n+1}\right)$, the partial sum is

$$\begin{aligned} s_n &= \frac{1}{2}\left(1 - \frac{1}{3}\right) + \frac{1}{2}\left(\frac{1}{3} - \frac{1}{5}\right) + \cdots \\ &\quad + \frac{1}{2}\left(\frac{1}{2n-3} - \frac{1}{2n-1}\right) + \frac{1}{2}\left(\frac{1}{2n-1} - \frac{1}{2n+1}\right) \\ &= \frac{1}{2}\left(1 - \frac{1}{2n+1}\right). \end{aligned}$$

Hence,

$$\sum_{n=1}^{\infty} \frac{1}{(2n-1)(2n+1)} = \lim s_n = \frac{1}{2}.$$

20. Since $1+2+3+\cdots+n = \dfrac{n(n+1)}{2}$, the given series is $\sum_{n=1}^{\infty} \dfrac{2}{n(n+1)}$ which converges to 2 by the result of Example 3 of this section.

21. The total distance is

$$2+2\left[2\times\frac{3}{4}+2\times\left(\frac{3}{4}\right)^2+\cdots\right]$$

$$=2+2\times\frac{3}{2}\left[1+\frac{3}{4}+\left(\frac{3}{4}\right)^2+\cdots\right]$$

$$=2+\frac{3}{1-\frac{3}{4}}=14 \text{ metres.}$$

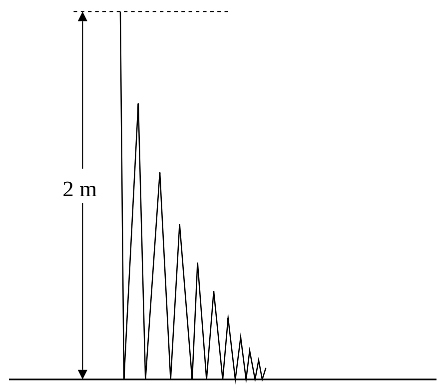

Fig. 2-21

30. "If $\sum a_n$ diverges and $\{b_n\}$ is bounded, then $\sum a_n\, b_n$ diverges" is FALSE. Let $a_n = \dfrac{1}{n}$ and $b_n = \dfrac{1}{n+1}$. Then $\sum a_n = \infty$ and $0 \le b_n \le 1/2$. But $\sum a_n b_n = \sum \dfrac{1}{n(n+1)}$ which converges by Example 3.

31. "If $a_n > 0$ and $\sum a_n$ converges, then $\sum a_n^2$ converges" is TRUE.
Since $\sum a_n$ converges, therefore $\lim a_n = 0$.
Thus there exists N such that $0 < a_n \le 1$ for $n \ge N$. Thus $0 < a_n^2 \le a_n$ for $n \ge N$.
If $S_n = \sum_{k=N}^{n} a_k^2$ and $s_n = \sum_{k=N}^{n} a_k$, then $\{S_n\}$ is increasing and bounded above:

$$S_n \le s_n \le \sum_{k=1}^{\infty} a_k < \infty.$$

Thus $\sum_{k=N}^{\infty} a_k^2$ converges, and so $\sum_{k=1}^{\infty} a_k^2$ converges.

27. a) "$\sum a_n$ converges implies $\sum(-1)^n a_n$ converges" is FALSE. $a_n = \dfrac{(-1)^n}{n}$ is a counterexample.

b) "$\sum a_n$ converges and $\sum(-1)^n a_n$ converges implies $\sum a_n$ converges absolutely" is FALSE. The series of Exercise 25 is a counterexample.

c) "$\sum a_n$ converges absolutely implies $\sum(-1)^n a_n$ converges absolutely" is TRUE, because
$|(-1)^n a_n| = |a_n|$.

41. Trying to apply the ratio test to $\sum \frac{2^{2n}(n!)^2}{(2n)!}$, we obtain

$$\rho = \lim \frac{2^{2n+2}((n+1)!)^2}{(2n+2)!} \cdot \frac{(2n)!}{2^{2n}(n!)^2} = \lim \frac{4(n+1)^2}{(2n+2)(2n+1)} = 1.$$

Thus the ratio test provides no information. However,

$$\begin{aligned} \frac{2^{2n}(n!)^2}{(2n)!} &= \frac{[2n(2n-2)\cdots 6 \cdot 4 \cdot 2]^2}{2n(2n-1)(2n-2)\cdots 3 \cdot 2 \cdot 1} \\ &= \frac{2n}{2n-1} \cdot \frac{2n-2}{2n-3} \cdot \cdots \cdot \frac{4}{3} \cdot \frac{2}{1} > 1. \end{aligned}$$

Since the terms exceed 1, the series diverges to infinity.

39. $\displaystyle\sum_{n=1}^{\infty}\left(\frac{n}{n+1}\right)^{n^2}$ converges by the root test of Exercise 31 since

$$\sigma = \lim_{n\to\infty}\left[\left(\frac{n}{n+1}\right)^{n^2}\right]^{1/n} = \lim_{n\to\infty}\frac{1}{\left(1+\frac{1}{n}\right)^{n}} = \frac{1}{e} < 1.$$

23. Apply the ratio test to $\sum \frac{(2x+3)^n}{n^{1/3}4^n}$:

$$\rho = \lim \left| \frac{(2x+3)^{n+1}}{(n+1)^{1/3}4^{n+1}} \cdot \frac{n^{1/3}4^n}{(2x+3)^n} \right| = \frac{|2x+3|}{4} = \frac{\left|x+\frac{3}{2}\right|}{2}.$$

The series converges absolutely if $\left|x+\frac{3}{2}\right| < 2$, that is, if $-\frac{7}{2} < x < \frac{1}{2}$. By the alternating series test it converges conditionally at $x = -\frac{7}{2}$. It diverges elsewhere.

24. Let $a_n = \frac{1}{n}\left(1 + \frac{1}{x}\right)^n$. Apply the ratio test

$$\rho = \lim \left| \frac{1}{n+1}\left(1 + \frac{1}{x}\right)^{n+1} \times \frac{n}{1}\left(1 + \frac{1}{x}\right)^{-n} \right| = \left|1 + \frac{1}{x}\right| < 1$$

if and only if $|x + 1| < |x|$, that is, $-2 < \frac{1}{x} < 0 \Rightarrow x < -\frac{1}{2}$. If $x = -\frac{1}{2}$, then $\sum_{n=1}^{\infty} a_n = \sum_{n=1}^{\infty} \frac{(-1)^n}{n}$, which converges conditionally. Thus, the series converges absolutely if $x < -\frac{1}{2}$, converges conditionally if $x = -\frac{1}{2}$ and diverges elsewhere. It is undefined at $x = 0$.

29. Applying the ratio test to $\sum \frac{(2n)!x^n}{2^{2n}(n!)^2} = \sum a_n x^n$, we obtain

$$\rho = \lim |x| \frac{(2n+2)(2n+1)}{4(n+1)^2} = |x|.$$

Thus $\sum a_n x^n$ converges absolutely if $-1 < x < 1$, and diverges if $x > 1$ or $x < -1$. In Exercise 36 of Section 9.3 it was shown that $a_n \geq \frac{1}{2n}$, so the given series definitely diverges at $x = 1$ and may at most converge conditionally at $x = -1$. To see whether it does converge at -1, we write, as in Exercise 36 of Section 9.3,

$$\begin{aligned} a_n &= \frac{(2n)!}{2^{2n}(n!)^2} = \frac{1 \times 2 \times 3 \times 4 \times \cdots \times 2n}{(2 \times 4 \times 6 \times 8 \times \cdots \times 2n)^2} \\ &= \frac{1 \times 3 \times 5 \times \cdots \times (2n-1)}{2 \times 4 \times 6 \times \cdots \times (2n-2) \times 2n} \\ &= \frac{1}{2} \times \frac{3}{4} \times \cdots \times \frac{2n-3}{2n-2} \times \frac{2n-1}{2n} \\ &= \left(1 - \frac{1}{2}\right)\left(1 - \frac{1}{4}\right) \cdots \left(1 - \frac{1}{2n-2}\right)\left(1 - \frac{1}{2n}\right). \end{aligned}$$

It is evident that a_n decreases as n increases. To see whether $\lim a_n = 0$, take logarithms and use the inequality $\ln(1+x) \leq x$:

$$\begin{aligned} \ln a_n &= \ln\left(1 - \frac{1}{2}\right) + \ln\left(1 - \frac{1}{4}\right) + \cdots + \ln\left(1 - \frac{1}{2n}\right) \\ &\leq -\frac{1}{2} - \frac{1}{4} - \cdots - \frac{1}{2n} \\ &= -\frac{1}{2}\left(1 + \frac{1}{2} + \cdots + \frac{1}{n}\right) \to -\infty \text{ as } n \to \infty. \end{aligned}$$

Thus $\lim a_n = 0$, and the given series converges conditionally at $x = -1$ by the alternating series test.