11. $$\int_0^1 \frac{dx}{\sqrt{x(1-x)}} = 2\int_0^{1/2} \frac{dx}{\sqrt{\frac{1}{4} - \left(x - \frac{1}{2}\right)^2}}$$

$$= 2 \lim_{c \to 0+} \int_c^{1/2} \frac{dx}{\sqrt{\frac{1}{4} - \left(x - \frac{1}{2}\right)^2}}$$

$$= 2 \lim_{c \to 0+} \sin^{-1}(2x-1)\Big|_c^{1/2} = \pi.$$

The integral converges.

14. $\displaystyle\int_0^{\pi/2} \sec x \, dx = \lim_{C \to (\pi/2)-} \ln|\sec x + \tan x| \Big|_0^C$

$\displaystyle = \lim_{C \to (\pi/2)-} \ln|\sec C + \tan C| = \infty.$

This integral diverges to infinity.

38. Since $0 \leq 1 - \cos\sqrt{x} = 2\sin^2\left(\frac{\sqrt{x}}{2}\right) \leq 2\left(\frac{\sqrt{x}}{2}\right)^2 = \frac{x}{2}$, for $x \geq 0$, therefore $\int_0^{\pi^2} \frac{dx}{1-\cos\sqrt{x}} \geq 2\int_0^{\pi^2} \frac{dx}{x}$, which diverges to infinity.

37. Since $\sin x \ge \dfrac{2x}{\pi}$ on $[0, \pi/2]$, we have

$$\begin{aligned}\int_0^\infty \frac{|\sin x|}{x^2}\,dx &\ge \int_0^{\pi/2} \frac{\sin x}{x^2}\,dx \\ &\ge \frac{2}{\pi}\int_0^{\pi/2} \frac{dx}{x} = \infty.\end{aligned}$$

The given integral diverges to infinity.

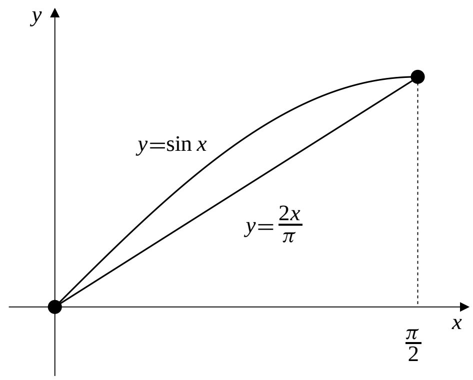

Fig. 5-37

40. Since $\ln x$ grows more slowly than any positive power of x, therefore we have $\ln x \le kx^{1/4}$ for some constant k and every $x \ge 2$. Thus, $\dfrac{1}{\sqrt{x}\ln x} \ge \dfrac{1}{kx^{3/4}}$ for $x \ge 2$ and $\displaystyle\int_2^\infty \frac{dx}{\sqrt{x}\ln x}$ diverges to infinity by comparison with $\displaystyle\frac{1}{k}\int_2^\infty \frac{dx}{x^{3/4}}$.

4. $\displaystyle\int_1^\infty \frac{dx}{x^2+\sqrt{x}+1}$ Let $x = \dfrac{1}{t^2}$

$$dx = -\frac{2\,dt}{t^3}$$

$$= \int_1^0 \frac{1}{\left(\frac{1}{t^2}\right)^2 + \sqrt{\frac{1}{t^2}} + 1}\left(-\frac{2\,dt}{t^3}\right)$$

$$= 2\int_0^1 \frac{t\,dt}{t^4+t^3+1}.$$

5. $\displaystyle\int_0^{\pi/2} \frac{dx}{\sqrt{\sin x}}$ Let $\sin x = u^2$

$2u\,du = \cos x\,dx = \sqrt{1-u^4}\,dx$

$$= 2\int_0^1 \frac{u\,du}{u\sqrt{1-u^4}}$$

$$= 2\int_0^1 \frac{du}{\sqrt{(1-u)(1+u)(1+y^2}}$$

Let $1-u = v^2$

$-du = 2v\,dv$

$$= 4\int_0^1 \frac{v\,dv}{v\sqrt{(1+1-v^2)(1+(1-v^2)^2)}}$$

$$= 4\int_0^1 \frac{dv}{\sqrt{(2-v^2)(2-2v^2+v^4)}}.$$

3. One possibility: let $x = \sin\theta$ and get

$$I = \int_{-1}^{1} \frac{e^x\, dx}{\sqrt{1-x^2}} = \int_{-\pi/2}^{\pi/2} e^{\sin\theta}\, d\theta.$$

Another possibility:

$$I = \int_{-1}^{0} \frac{e^x\, dx}{\sqrt{1-x^2}} + \int_{0}^{1} \frac{e^x\, dx}{\sqrt{1-x^2}} = I_1 + I_2.$$

In I_1 put $1 + x = u^2$; in I_2 put $1 - x = u^2$:

$$I_1 = \int_0^1 \frac{2e^{u^2-1} u\, du}{u\sqrt{2-u^2}} = 2\int_0^1 \frac{e^{u^2-1}\, du}{\sqrt{2-u^2}}$$
$$I_2 = \int_0^1 \frac{2e^{1-u^2} u\, du}{u\sqrt{2-u^2}} = 2\int_0^1 \frac{e^{1-u^2}\, du}{\sqrt{2-u^2}}$$

so $I = 2\displaystyle\int_0^1 \frac{e^{u^2-1} + e^{1-u^2}}{\sqrt{2-u^2}}\, du.$

8. Let

$$I = \int_1^\infty e^{-x^2}\,dx \quad \text{Let } x = \frac{1}{t}$$

$$dx = -\frac{dt}{t^2}$$

$$= \int_1^0 e^{-(1/t)^2}\left(-\frac{1}{t^2}\right)dt = \int_0^1 \frac{e^{-1/t^2}}{t^2}\,dt.$$

Observe that

$$\lim_{t\to 0+}\frac{e^{-1/t^2}}{t^2} = \lim_{t\to 0+}\frac{t^{-2}}{e^{1/t^2}} \quad \left[\frac{\infty}{\infty}\right]$$

$$= \lim_{t\to 0+}\frac{-2t^{-3}}{e^{1/t^2}(-2t^{-3})}$$

$$= \lim_{t\to 0+}\frac{1}{e^{1/t^2}} = 0.$$

Hence,

$$S_2 = \frac{1}{3}\left(\frac{1}{2}\right)\left[0 + 4(4e^{-4}) + e^{-1}\right]$$

$$\approx 0.1101549$$

$$S_4 = \frac{1}{3}\left(\frac{1}{4}\right)\Big[0 + 4(16e^{-16}) + 2(4e^{-4})$$

$$+ 4\Big(\frac{16}{9}e^{-16/9}\Big) + e^{-1}\Big]$$

$$\approx 0.1430237$$

$$S_8 = \frac{1}{3}\left(\frac{1}{8}\right)\Big[0 + 4\Big(64e^{-64} + \frac{64}{9}e^{-64/9} + \frac{64}{25}e^{-64/25} +$$

$$\frac{64}{49}e^{-64/49}\Big) + 2\Big(16e^{-16} + 4e^{-4} + \frac{16}{9}e^{-16/9}\Big) + e^{-1}\Big]$$

$$\approx 0.1393877.$$

Hence, $I \approx 0.14$, accurate to 2 decimal places. These approximations do not converge very quickly, because the fourth derivative of e^{-1/t^2} has very large values for some values of t near 0. In fact, higher and higher derivatives behave more and more badly near 0, so higher order methods cannot be expected to work well either.

14. Let $y = f(x)$. We are given that m_1 is the midpoint of $[x_0, x_1]$ where $x_1 - x_0 = h$. By tangent line approximate in the subinterval $[x_0, x_1]$,

$$f(x) \approx f(m_1) + f'(m_1)(x - m_1).$$

The error in this approximation is

$$E(x) = f(x) - f(m_1) - f'(m_1)(x - m_1).$$

If $f''(t)$ exists for all t in $[x_0, x_1]$ and $|f''(t)| \le K$ for some constant K, then by Theorem 11 of Section 4.9,

$$|E(x)| \le \frac{K}{2}(x - m_1)^2.$$

Hence,

$$|f(x) - f(m_1) - f'(m_1)(x - m_1)| \le \frac{K}{2}(x - m_1)^2.$$

We integrate both sides of this inequlity. Noting that $x_1 - m_1 = m_1 - x_0 = \frac{1}{2}h$, we obtain for the left side

$$\begin{aligned}
&\left|\int_{x_0}^{x_1} f(x)\,dx - \int_{x_0}^{x_1} f(m_1)\,dx \right. \\
&\qquad \left. - \int_{x_0}^{x_1} f'(m_1)(x - m_1)\,dx\right| \\
&= \left|\int_{x_0}^{x_1} f(x)\,dx - f(m_1)h - f'(m_1)\frac{(x - m_1)^2}{2}\bigg|_{x_0}^{x_1}\right| \\
&= \left|\int_{x_0}^{x_1} f(x)\,dx - f(m_1)h\right|.
\end{aligned}$$

Integrating the right-hand side, we get

$$\begin{aligned}
\int_{x_0}^{x_1} \frac{K}{2}(x - m_1)^2\,dx &= \frac{K}{2}\frac{(x - m_1)^3}{3}\bigg|_{x_0}^{x_1} \\
&= \frac{K}{6}\left(\frac{h^3}{8} + \frac{h^3}{8}\right) = \frac{K}{24}h^3.
\end{aligned}$$

Hence,

$$\begin{aligned}
&\left|\int_{x_0}^{x_1} f(x)\,dx - f(m_1)h\right| \\
&= \left|\int_{x_0}^{x_1} [f(x) - f(m_1) - f'(m_1)(x - m_1)]\,dx\right| \\
&\le \frac{K}{24}h^3.
\end{aligned}$$

13. $I = \int_0^1 x^2\,dx = \frac{1}{3}$. $M_1 = \left(\frac{1}{2}\right)^2 (1) = \frac{1}{4}$. The actual error is $I - M_1 = \frac{1}{3} - \frac{1}{4} = \frac{1}{12}$.
Since the second derivative of x^2 is 2, the error estimate is

$$|I - M_1| \le \frac{2}{24}(1-0)^2(1^2) = \frac{1}{12}.$$

Thus the constant in the error estimate for the Midpoint Rule cannot be improved; no smaller constant will work for $f(x) = x^2$.

46. $\Gamma(x) = \int_0^\infty t^{x-1}e^{-t}\,dt.$

a) Since $\lim_{t\to\infty} t^{x-1}e^{-t/2} = 0$, there exists $T > 0$ such that $t^{x-1}e^{-t/2} \le 1$ if $t \ge T$. Thus

$$0 \le \int_T^\infty t^{x-1}e^{-t}\,dt \le \int_T^\infty e^{-t}\,dt = 2e^{-T/2}$$

and $\int_T^\infty t^{x-1}e^{-t}\,dt$ converges by the comparison theorem.

If $x > 0$, then

$$0 \le \int_0^T t^{x-1}e^{-t}\,dt < \int_0^T t^{x-1}\,dt$$

converges by Theorem 2(b). Thus the integral defining $\Gamma(x)$ converges.

b)
$$\begin{aligned}
\Gamma(x+1) &= \int_0^\infty t^x e^{-t}\,dt \\
&= \lim_{\substack{c\to 0+\\ R\to\infty}} \int_c^R t^x e^{-t}\,dt \\
&\qquad U = t^x \qquad\qquad dV = e^{-t}\,dt \\
&\qquad dU = xt^{x-1}\,dx \qquad V = -e^{-t} \\
&= \lim_{\substack{c\to 0+\\ R\to\infty}} \left(-t^x e^{-t}\Big|_c^R + x\int_c^R t^{x-1}e^{-t}\,dt\right) \\
&= 0 + x\int_0^\infty t^{x-1}e^{-t}\,dt = x\Gamma(x).
\end{aligned}$$

c) $\Gamma(1) = \int_0^\infty e^{-t}\,dt = 1 = 0!.$
By (b), $\Gamma(2) = 1\Gamma(1) = 1 \times 1 = 1 = 1!.$
In general, if $\Gamma(k+1) = k!$ for some positive integer k, then
$\Gamma(k+2) = (k+1)\Gamma(k+1) = (k+1)k! = (k+1)!.$
Hence $\Gamma(n+1) = n!$ for all integers $n \ge 0$, by induction.

d)
$$\begin{aligned}
\Gamma\left(\frac{1}{2}\right) &= \int_0^\infty t^{-1/2}e^{-t}\,dt \qquad \text{Let } t = x^2 \\
&\qquad\qquad\qquad\qquad\quad dt = 2x\,dx \\
&= \int_0^\infty \frac{1}{x}e^{-x^2}\,2x\,dx = 2\int_0^\infty e^{-x^2}\,dx = \sqrt{\pi}
\end{aligned}$$

$$\Gamma\left(\frac{3}{2}\right) = \frac{1}{2}\Gamma\left(\frac{1}{2}\right) = \frac{1}{2}\sqrt{\pi}.$$