21. $\displaystyle\int \frac{\ln(\ln x)}{x}\,dx$ Let $u = \ln x$

$$du = \frac{dx}{x}$$

$$= \int \ln u\,du$$

$$U = \ln u \quad dV = du$$

$$dU = \frac{du}{u} \quad V = u$$

$$= u\ln u - \int du = u\ln u - u + C$$

$$= (\ln x)(\ln(\ln x)) - \ln x + C.$$

26. $\int (\sin^{-1} x)^2\,dx$ Let $x = \sin\theta$

$dx = \cos\theta\,d\theta$

$= \int \theta^2 \cos\theta\,d\theta$

$U = \theta^2 \qquad dV = \cos\theta\,d\theta$

$dU = 2\theta\,d\theta \qquad V = \sin\theta$

$= \theta^2 \sin\theta - 2\int \theta \sin\theta\,d\theta$

$U = \theta \qquad dV = \sin\theta\,d\theta$

$dU = d\theta \qquad V = -\cos\theta$

$= \theta^2 \sin\theta - 2(-\theta\cos\theta + \int \cos\theta\,d\theta)$

$= \theta^2 \sin\theta + 2\theta\cos\theta - 2\sin\theta + C$

$= x(\sin^{-1} x)^2 + 2\sqrt{1-x^2}(\sin^{-1} x) - 2x + C.$

28. By the procedure used in Example 4 of Section 7.1,

$$\int e^x \cos x\, dx = \tfrac{1}{2}e^x(\sin x + \cos x) + C;$$
$$\int e^x \sin x\, dx = \tfrac{1}{2}e^x(\sin x - \cos x) + C.$$

Now

$$\begin{aligned}
&\int xe^x \cos x\, dx \\
&\qquad U = x \qquad dV = e^x \cos x\, dx \\
&\qquad dU = dx \qquad V = \tfrac{1}{2}e^x(\sin x + \cos x) \\
&= \tfrac{1}{2}xe^x(\sin + \cos x) - \tfrac{1}{2}\int e^x(\sin x + \cos x)\, dx \\
&= \tfrac{1}{2}xe^x(\sin + \cos x) \\
&\qquad - \tfrac{1}{4}e^x(\sin x - \cos x + \sin x + \cos x) + C \\
&= \tfrac{1}{2}xe^x(\sin x + \cos x) - \tfrac{1}{2}e^x \sin x + C.
\end{aligned}$$

16. First divide to obtain

$$\begin{aligned}
\frac{x^3+1}{x^2+7x+12} &= x-7+\frac{37x+85}{(x+4)(x+3)} \\
\frac{37x+85}{(x+4)(x+3)} &= \frac{A}{x+4}+\frac{B}{x+3} \\
&= \frac{(A+B)x+3A+4B}{x^2+7x+12} \\
&\Rightarrow \begin{cases} A+B=37 \\ 3A+4B=85 \end{cases} \Rightarrow A=63,\ \ B=-26.
\end{aligned}$$

Now we have

$$\begin{aligned}
&\int \frac{x^3+1}{12+7x+x^2}\,dx = \int\left(x-7+\frac{63}{x+4}-\frac{26}{x+3}\right)dx \\
&= \frac{x^2}{2}-7x+63\ln|x+4|-26\ln|x+3|+C.
\end{aligned}$$

26. We have

$$\int \frac{dt}{(t-1)(t^2-1)^2}$$
$$= \int \frac{dt}{(t-1)^3(t+1)^2} \quad \text{Let } u = t-1$$
$$du = dt$$
$$= \int \frac{du}{u^3(u+2)^2}$$
$$\frac{1}{u^3(u+2)^2} = \frac{A}{u} + \frac{B}{u^2} + \frac{C}{u^3} + \frac{D}{u+2} + \frac{E}{(u+2)^2}$$
$$= \frac{A(u^4+4u^3+4u^2) + B(u^3+4u^2+4u)}{u^3(u+2)^2}$$
$$\frac{C(u^2+4u+4) + D(u^4+2u^3) + Eu^3}{u^3(u+2)^2}$$
$$\Rightarrow \begin{cases} A + D = 0 \\ 4A + B + 2D + E = 0 \\ 4A + 4B + C = 0 \\ 4B + 4C = 0 \\ 4C = 1 \end{cases}$$
$$\Rightarrow A = \frac{3}{16},\ B = -\frac{1}{4},\ C = \frac{1}{4},\ D = -\frac{3}{16},\ E = -\frac{1}{8}.$$
$$\int \frac{du}{u^3(u+2)^2}$$
$$= \frac{3}{16}\int \frac{du}{u} - \frac{1}{4}\int \frac{du}{u^2} + \frac{1}{4}\int \frac{du}{u^3}$$
$$- \frac{3}{16}\int \frac{du}{u+2} - \frac{1}{8}\int \frac{du}{(u+2)^2}$$
$$= \frac{3}{16}\ln|t-1| + \frac{1}{4(t-1)} - \frac{1}{8(t-1)^2} -$$
$$\frac{3}{16}\ln|t+1| + \frac{1}{8(t+1)} + K.$$

27. $\displaystyle\int \frac{dx}{e^{2x} - 4e^{x} + 4} = \int \frac{dx}{(e^{x} - 2)^2}$ Let $u = e^{x}$

$du = e^{x}\,dx$

$$= \int \frac{du}{u(u-2)^2}$$

$$\frac{1}{u(u-2)^2} = \frac{A}{u} + \frac{B}{u-2} + \frac{C}{(u-2)^2}$$

$$= \frac{A(u^2 - 4u + 4) + B(u^2 - 2u) + +Cu}{u(u-2)^2}$$

$$\Rightarrow \begin{cases} A + B = 0 \\ -4A - 2B + C = 0 \\ 4A = 1 \end{cases} \Rightarrow A = \frac{1}{4},\ B = -\frac{1}{4},\ C = \frac{1}{2}.$$

$$\int \frac{du}{u(u-2)^2} = \frac{1}{4}\int \frac{du}{u} - \frac{1}{4}\int \frac{du}{u-2} + \frac{1}{2}\int \frac{du}{(u-2)^2}$$

$$= \frac{1}{4}\ln|u| - \frac{1}{4}\ln|u-2| - \frac{1}{2}\frac{1}{(u-2)} + K$$

$$= \frac{x}{4} - \frac{1}{4}\ln|e^{x} - 2| - \frac{1}{2(e^{x} - 2)} + K.$$

40. $\displaystyle\int \frac{dx}{x^2(x^2-1)^{3/2}}$ Let $x = \sec\theta$

$dx = \sec\theta \tan\theta \, d\theta$

$$= \int \frac{\sec\theta \tan\theta \, d\theta}{\sec^2\theta \tan^3\theta} = \int \frac{\cos^3\theta \, d\theta}{\sin^2\theta}$$

$$= \int \frac{1-\sin^2\theta}{\sin^2\theta} \cos\theta \, d\theta$$ Let $u = \sin\theta$

$du = \cos\theta \, d\theta$

$$= \int \frac{1-u^2}{u^2} \, du = -\frac{1}{u} - u + C$$

$$= -\left(\frac{1}{\sin\theta} + \sin\theta\right) + C$$

$$= -\left(\frac{x}{\sqrt{x^2-1}} + \frac{\sqrt{x^2-1}}{x}\right) + C.$$

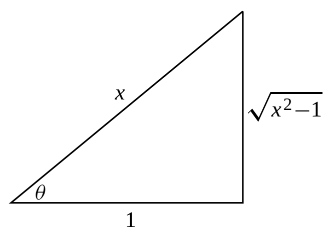

Fig. 3-40

41. $I=\int \frac{dx}{x(1+x^2)^{3/2}}$ Let $x=\tan\theta$
$dx=\sec^2\theta\,d\theta$

$$=\int \frac{\sec^2\theta\,d\theta}{\tan\theta\sec^3\theta}=\int \frac{\cos^2\theta\,d\theta}{\sin\theta}$$

$$=\int \frac{\cos^2\theta\sin\theta\,d\theta}{\sin^2\theta}$$ Let $u=\cos\theta$
$du=-\sin\theta\,d\theta$

$$=-\int \frac{u^2\,du}{1-u^2}=u+\int \frac{du}{u^2-1}.$$

We have

$$\frac{1}{u^2-1}=\frac{1}{2}\left(\frac{1}{u-1}-\frac{1}{u+1}\right).$$

Thus

$$\begin{aligned} I &= u+\frac{1}{2}\ln\left|\frac{u-1}{u+1}\right|+C \\ &= \cos\theta+\frac{1}{2}\ln\left|\frac{\cos\theta-1}{\cos\theta+1}\right|+C \\ &= \frac{1}{\sqrt{1+x^2}}+\frac{1}{2}\ln\left|\frac{\frac{1}{\sqrt{1+x^2}}-1}{\frac{1}{\sqrt{1+x^2}}+1}\right|+C \\ &= \frac{1}{\sqrt{1+x^2}}+\frac{1}{2}\ln\left(\frac{\sqrt{1+x^2}-1}{\sqrt{1+x^2}+1}\right)+C. \end{aligned}$$

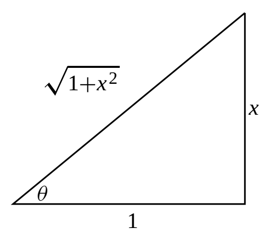

Fig. 3-41

44. $\displaystyle\int_0^{\pi/2} \frac{d\theta}{1+\cos\theta+\sin\theta}$ Let $x = \tan\dfrac{\theta}{2}$, $d\theta = \dfrac{2}{1+x^2}\,dx$,

$\cos\theta = \dfrac{1-x^2}{1+x^2}$, $\sin\theta = \dfrac{2x}{1+x^2}$.

$$= \int_0^1 \frac{\left(\frac{2}{1+x^2}\right) dx}{1+\left(\frac{1-x^2}{1+x^2}\right)+\left(\frac{2x}{1+x^2}\right)}$$

$$= 2\int_0^1 \frac{dx}{2+2x} = \int_0^1 \frac{dx}{1+x}$$

$$= \ln|1+x|\Big|_0^1 = \ln 2.$$

4. As an alternative to the direct method of Example 3, we begin with the change of variable $u = \ln x$, or, equivalently, $x = e^u$, so that $dx = e^u\, du$.

$$
\begin{aligned}
I &= \int x^2 (\ln x)^4\, dx = \int u^4\, e^{3u}\, du \\
&= \left(a_4 u^4 + a_3 u^3 + a_2 u^2 + a_1 u + a_0\right)\, e^{3u} + C.
\end{aligned}
$$

We will have

$$
\begin{aligned}
\frac{dI}{du} &= \left(4a_4 u^3 + 3a_3 u^2 + 2a_2 u + a_1\right) e^{3u} \\
&\quad + 3\left(a_4 u^4 + a_3 u^3 + a_2 u^2 + a_1 u + a_0\right) e^{3u} \\
&= u^4\, e^{3u}
\end{aligned}
$$

provided $3a_4 = 1$, $3a_3 + 4a_4 = 0$, $3a_2 + 3a_3 = 0$, $3a_1 + 2a_2 = 0$, and $3a_0 + a_1 = 0$. Thus $a_4 = 1/3$, $a_3 = -4/9$, $a_2 = 4/9$, $a_1 = -8/27$, and $a_0 = 8/81$. We now have

$$
\begin{aligned}
I &= \left(\frac{1}{3}u^4 - \frac{4}{9}u^3 + \frac{4}{9}u^2 - \frac{8}{27}u + \frac{8}{81}\right) e^{3u} + C \\
&\int x^2 (\ln x)^4\, dx \\
&= x^3 \left(\frac{(\ln x)^4}{3} - \frac{4(\ln x)^3}{9} + \frac{4(\ln x)^2}{9} - \frac{8 \ln x}{27} + \frac{8}{81}\right) + C
\end{aligned}
$$

29. Since $(x-1)(x^2-1)(x^3-1) = (x-1)^3(x+1)(x^2+x+1)$, and the numerator has degree less than the denominator, we have

$$\frac{x^5+x^3+1}{(x-1)(x^2-1)(x^3-1)} = \frac{A}{x-1} + \frac{B}{(x-1)^2} + \frac{3}{(x-1)^3} + \frac{D}{x+1} + \frac{Ex+F}{x^2+x+1}.$$

19. Suppose that $I = \int e^{-x^2}\,dx = P(x)\,e^{-x^2} + C$, where P is a polynomial having, say, degree $m \geq 0$.

(a) Then we must have

$$\frac{dI}{dx} = \big(P'(x) - 2xP(x)\big)\,e^{-x^2} = e^{-x^2}.$$

It follows that $P'(x) - 2xP(x) = 1$. The left side of this equation must be a polynomial of degree $m + 1 \geq 1$ because $2xP(x)$ has degree $m + 1$ and $P'(x)$ only has degree $m - 1$. But the right side of the equation is a polynomial of degree 0 (i.e., a constant). This contradiction shows that no such polynomial $P(x)$ can exist.

(b) Since $\dfrac{d}{dx}\,\mathrm{Erf}(x) = \dfrac{2}{\sqrt{\pi}}\,e^{-x^2}$ by the Fundamental Theorem of Calculus, we have

$$\int e^{-x^2}\,dx = \frac{\sqrt{\pi}}{2}\mathrm{Erf}(x) + C.$$

(c) Let us try the form

$$J = \int \mathrm{Erf}(x)\,dx = P(x)\mathrm{Erf}(x) + Q(x)\,e^{-x^2} + C,$$

where P and Q are polynomials to be determined. Then

$$\begin{aligned}\frac{dJ}{dx} &= P'(x)\,\mathrm{Erf}(x) \\ &\qquad + \left(\frac{2}{\sqrt{\pi}}P(x) + Q'(x) - 2xQ(x)\right)\,e^{-x^2} \\ &= \mathrm{Erf}(x).\end{aligned}$$

Hence we must have $P'(x) = 1$ and $\dfrac{2}{\sqrt{\pi}}P(x) + Q'(x) - 2xQ(x) = 0$. The first of these DEs says that $P(x) = x + k$; without loss of generality we can take the constant k to be zero. The second DE says that

$$Q'(x) - 2xQ(x) = -\frac{2x}{\sqrt{\pi}}.$$

The right side has degree 1 and so must the left side. Thus Q must have degree zero. Hence $Q'(x) = 0$ and $Q(x) = 1/\sqrt{\pi}$. Therefore

$$J = \int \mathrm{Erf}(x)\,dx = x\mathrm{Erf}(x) + \frac{1}{\sqrt{\pi}}\,e^{-x^2} + C.$$

32. $I_n = \int_0^{\pi/2} x^n \sin x\, dx$

$$U = x^n \qquad dV = \sin x\, dx$$

$$dU = nx^{n-1}\, dx \qquad V = -\cos x$$

$$= -x^n \cos x\Big|_0^{\pi/2} + n\int_0^{\pi/2} x^{n-1} \cos x\, dx$$

$$U = x^{n-1} \qquad dV = \cos x\, dx$$

$$dU = (n-1)x^{n-2}\, dx \qquad V = \sin x$$

$$= n\left[x^{n-1} \sin x\Big|_0^{\pi/2} - (n-1)\int_0^{\pi/2} x^{n-2} \sin x\, dx\right]$$

$$= n\left(\frac{\pi}{2}\right)^{n-1} - n(n-1)I_{n-2}, \qquad (n \ge 2).$$

$$I_0 = \int_0^{\pi/2} \sin x\, dx = -\cos x\Big|_0^{\pi/2} = 1.$$

$$I_6 = 6\left(\frac{\pi}{2}\right)^5 - 6(5)\left\{4\left(\frac{\pi}{2}\right)^3 - 4(3)\left[2\left(\frac{\pi}{2}\right) - 2(1)I_0\right]\right\}$$

$$= \frac{3}{16}\pi^5 - 15\pi^3 + 360\pi - 720.$$

33. $I_n = \int \sin^n x\,dx \quad (n \ge 2)$

$$\begin{aligned} U &= \sin^{n-1} x & dV &= \sin x\,dx \\ dU &= (n-1)\sin^{n-2} x \cos x\,dx & V &= -\cos x \end{aligned}$$

$$\begin{aligned} &= -\sin^{n-1} x \cos x + (n-1)\int \sin^{n-2} x \cos^2 x\,dx \\ &= -\sin^{n-1} x \cos x + (n-1)(I_{n-2} - I_n) \\ nI_n &= -\sin^{n-1} x \cos x + (n-1)I_{n-2} \\ I_n &= -\frac{1}{n}\sin^{n-1} x \cos x + \frac{n-1}{n} I_{n-2}. \end{aligned}$$

Note: $I_0 = x + C$, $I_1 = -\cos x + C$. Hence

$$\begin{aligned} I_6 &= -\frac{1}{6}\sin^5 x \cos x + \frac{5}{6} I_4 \\ &= -\frac{1}{6}\sin^5 x \cos x + \frac{5}{6}\left(-\frac{1}{4}\sin^3 x \cos x + \frac{3}{4} I_2\right) \\ &= -\frac{1}{6}\sin^5 x \cos x - \frac{5}{24}\sin^3 x \cos x \\ &\qquad + \frac{5}{8}\left(-\frac{1}{2}\sin x \cos x + \frac{1}{2} I_0\right) \\ &= -\frac{1}{6}\sin^5 x \cos x - \frac{5}{24}\sin^3 x \cos x - \frac{5}{16}\sin x \cos x \\ &\qquad + \frac{5}{16}x + C \\ &= \frac{5x}{16} - \cos x\left(\frac{\sin^5 x}{6} + \frac{5\sin^3 x}{24} + \frac{5\sin x}{16}\right) + C. \end{aligned}$$

$$\begin{aligned} I_7 &= -\frac{1}{7}\sin^6 x \cos x + \frac{6}{7} I_5 \\ &= -\frac{1}{7}\sin^6 x \cos x + \frac{6}{7}\left(-\frac{1}{5}\sin^4 x \cos x + \frac{4}{5} I_3\right) \\ &= -\frac{1}{7}\sin^6 x \cos x - \frac{6}{35}\sin^4 x \cos x \\ &\qquad + \frac{24}{35}\left(-\frac{1}{3}\sin^2 x \cos x + \frac{2}{3} I_1\right) \\ &= -\frac{1}{7}\sin^6 x \cos x - \frac{6}{35}\sin^4 x \cos x - \frac{8}{35}\sin^2 x \cos x \\ &\qquad - \frac{16}{35}\cos x + C \\ &= -\cos x\left(\frac{\sin^6 x}{7} + \frac{6\sin^4 x}{35} + \frac{8\sin^2 x}{35} + \frac{16}{35}\right) + C. \end{aligned}$$