9. $2^2 - 3^2 + 4^2 - 5^2 + \cdots - 99^2 = \sum_{i=2}^{99} (-1)^i i^2$

14. $\frac{1}{2} + \frac{2}{4} + \frac{3}{8} + \frac{4}{16} + \cdots + \frac{n}{2^n} = \sum_{i=1}^{n} \frac{i}{2^i}$

41. The formula $\sum_{i=1}^{n} i^3 = n^2(n+1)^2/4$ holds for $n = 1$, since it says $1 = 1$ in this case. Now assume that it holds for $n =$ some number $k \geq 1$; that is, $\sum_{i=1}^{k} i^3 = k^2(k+1)^2/4$. Then for $n = k+1$, we have

$$\begin{aligned}
\sum_{i=1}^{k+1} i^3 &= \sum_{i=1}^{k} i^3 + (k+1)^3 \\
&= \frac{k^2(k+1)^2}{4} + (k+1)^3 = \frac{(k+1)^2}{4}[k^2 + 4(k+1)] \\
&= \frac{(k+1)^2}{4}(k+2)^2.
\end{aligned}$$

Thus the formula also holds for $n = k+1$. By induction, it holds for all positive integers n.

7. The required area is (see the figure)

$$
\begin{aligned}
A = \lim_{n\to\infty} \frac{3}{n}\Bigg[&\left(-1+\frac{3}{n}\right)^2 + 2\left(-1+\frac{3}{n}\right) + 3 \\
&+ \left(-1+\frac{6}{n}\right)^2 + 2\left(-1+\frac{6}{n}\right) + 3 \\
&+ \cdots + \left(-1+\frac{3n}{n}\right)^2 + 2\left(-1+\frac{3n}{n}\right) + 3\Bigg] \\
= \lim_{n\to\infty} \frac{3}{n}\Bigg[&\left(1-\frac{6}{n}+\frac{3^2}{n^2}-2+\frac{6}{n}+3\right) \\
&+ \left(1-\frac{12}{n}+\frac{6^2}{n^2}-2+\frac{12}{n}+3\right) \\
&+ \cdots + \left(1-\frac{6n}{n}+\frac{9n^2}{n^2}-2+\frac{6n}{n}+3\right)\Bigg] \\
= \lim_{n\to\infty} &\left(6+\frac{27}{n^3}\cdot\frac{n(n+1)(2n+1)}{6}\right) \\
= 6 + 9 = 15&\text{sq. units.}
\end{aligned}
$$

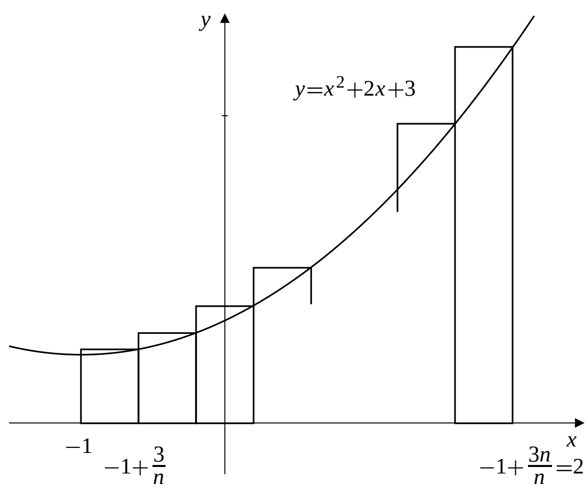

Fig. 2-7

10.

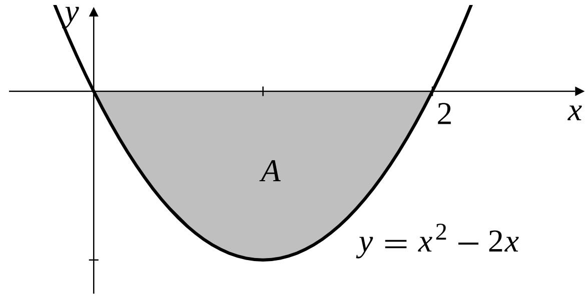

Fig. 2-10

The height of the region at position x is $0 - (x^2 - 2x) = 2x - x^2$. The "base" is an interval of length 2, so we approximate using n rectangles of width $2/n$. The shaded area is

$$\begin{aligned}
A &= \lim_{n\to\infty} \sum_{i=1}^{n} \frac{2}{n} \left(2\frac{2i}{n} - \frac{4i^2}{n^2} \right) \\
&= \lim_{n\to\infty} \sum_{i=1}^{n} \left(\frac{8i}{n^2} - \frac{8i^2}{n^3} \right) \\
&= \lim_{n\to\infty} \left(\frac{8}{n^2} \, \frac{n(n+1)}{2} - \frac{8}{n^3} \, \frac{n(n+1)(2n+1)}{6} \right) \\
&= 4 - \frac{8}{3} = \frac{4}{3} \text{ sq. units.}
\end{aligned}$$

18. $P = \{x_0 < x_1 < \cdots < x_n\}$,
$P' = \{x_0 < x_1 < \cdots < x_{j-1} < x' < x_j < \cdots < x_n\}$.
Let m_i and M_i be, respectively, the minimum and maximum values of $f(x)$ on the interval $[x_{i-1}, x_i]$, for $1 \le i \le n$. Then

$$L(f, P) = \sum_{i=1}^{n} m_i (x_i - x_{i-1}),$$
$$U(f, P) = \sum_{i=1}^{n} M_i (x_i - x_{i-1}).$$

If m_j' and M_j' are the minimum and maximum values of $f(x)$ on $[x_{j-1}, x']$, and if m_j'' and M_j'' are the corresponding values for $[x', x_j]$, then

$$m_j' \ge m_j, \quad m_j'' \ge m_j, \quad M_j' \le M_j, \quad M_j'' \le M_j.$$

Therefore we have

$$m_j(x_j - x_{j-1}) \le m_j'(x' - x_{j-1}) + m_j''(x_j - x'),$$
$$M_j(x_j - x_{j-1}) \ge M_j'(x' - x_{j-1}) + M_j''(x_j - x').$$

Hence $L(f, P) \le L(f, P')$ and $U(f, P) \ge U(f, P')$.

If P'' is any refinement of P we can add the new points in P'' to those in P one at a time, and thus obtain

$$L(f, P) \le L(f, P''), \qquad U(f, P'') \le U(f, P).$$

18.

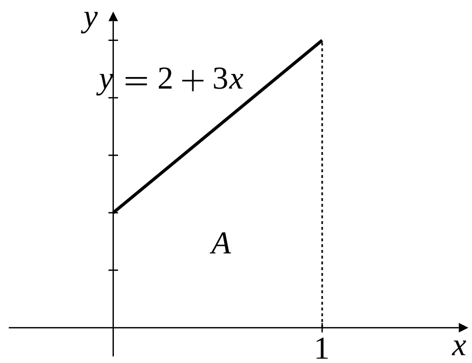

Fig. 2-18

$s_n = \sum_{i=1}^{n} \frac{2n+3i}{n^2} = \sum_{i=1}^{n} \frac{1}{n}\left(2 + \frac{3i}{n}\right)$ represents a sum of areas of n rectangles each of width $1/n$ and having heights equal to the height to the graph $y = 2 + 3x$ at the points $x = i/n$. Thus $\lim_{n\to\infty} S_n$ is the area of the trapezoid in the figure above, and has the value $1(2+5)/2 = 7/2$.

14. $$\int_{-3}^{3}(2+t)\sqrt{9-t^2}\,dt = 2\int_{-3}^{3}\sqrt{9-t^2}\,dt + \int_{-3}^{3}t\sqrt{9-t^2}\,dt$$
$$= 2\left(\frac{1}{2}\pi 3^2\right) + 0 = 9\pi$$

33. $\int_{-1}^{2} \operatorname{sgn} x \, dx = 2 - 1 = 1$

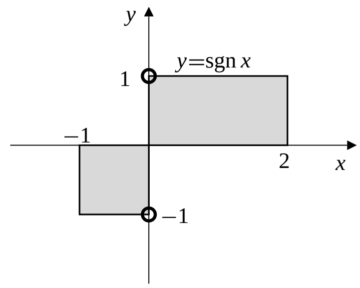

Fig. 4-33

39.

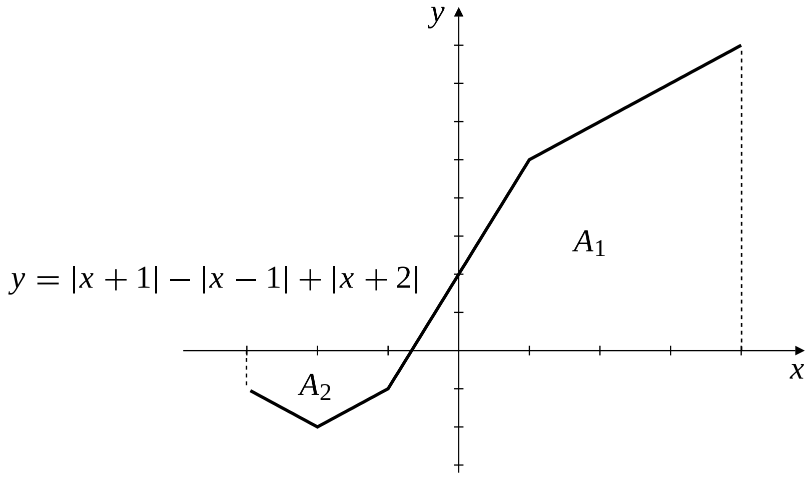

Fig. 4-39

$$\int_{-3}^{4} (|x+1| - |x-1| + |x+2|)\,dx$$
$$= \text{area } A_1 - \text{area } A_2$$
$$= \frac{1}{2}\,\frac{5}{3}(5) + \frac{5+8}{2}(3) - \frac{1+2}{2}(1) - \frac{1+2}{2}(1) - \frac{1}{2}\,\frac{1}{3}(1) = \frac{41}{2}$$

40.

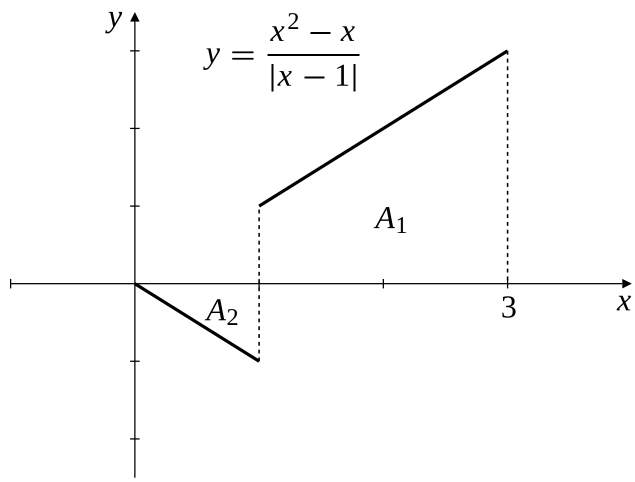

Fig. 4-40

$$\int_0^3 \frac{x^2 - x}{|x-1|}\,dx$$
$$= \text{area } A_1 - \text{area } A_2$$
$$= \frac{1+3}{2}(2) - \frac{1}{2}(1)(1) = \frac{7}{2}$$

43. $\int_a^b \left(f(x) - k\right)^2 dx$

$$= \int_a^b \left(f(x)\right)^2 dx - 2k \int_a^b f(x)\, dx + k^2 \int_a^b dx$$

$$= \int_a^b \left(f(x)\right)^2 dx - 2k(b-a)\overline{f} + k^2(b-a)$$

$$= (b-a)(k-\overline{f})^2 + \int_a^b \left(f(x)\right)^2 dx - (b-a)\overline{f}^2$$

This is minimum if $k = \overline{f}$.