1. If $a_n > 0$ and $\dfrac{a_{n+1}}{a_n} > \dfrac{n}{n+1}$ for all n, then

$$\begin{aligned}
\frac{a_2}{a_1} > \frac{1}{2} \quad &\Rightarrow \quad a_2 > \frac{a_1}{2} \\
\frac{a_3}{a_2} > \frac{2}{3} \quad &\Rightarrow \quad a_3 > \frac{2a_2}{3} > \frac{a_1}{3} \\
&\vdots \\
\frac{a_n}{a_{n-1}} > \frac{n-1}{n} \quad &\Rightarrow \quad a_n > \frac{a_1}{n}.
\end{aligned}$$

(This can be verified by induction.)

Therefore $\sum_{n=1}^{\infty} a_n$ diverges by comparison with the harmonic series $\sum_{n=1}^{\infty} \dfrac{1}{n}$.

45. $$\begin{aligned} S(x) &= \int_0^x \sin(t^2)\,dt \\ &= \int_0^x \left(t^2 - \frac{t^6}{3!} + \cdots\right) dt \\ &= \frac{x^3}{3} - \frac{x^7}{7\cdot 3!} + \cdots \end{aligned}$$

$$\lim_{x\to 0} \frac{x^3 - 3S(x)}{x^7} = \lim_{x\to 0} \frac{x^3 - x^3 + \dfrac{x^7}{14} - \cdots}{x^7} = \frac{1}{14}.$$

38. $$\sqrt{1+\sin x} = 1 + \frac{1}{2}\sin x + \frac{\left(\frac{1}{2}\right)\left(-\frac{1}{2}\right)}{2!}(\sin x)^2 + \frac{\left(\frac{1}{2}\right)\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)}{3!}(\sin x)^3 + \frac{\left(\frac{1}{2}\right)\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)\left(-\frac{5}{2}\right)}{4!}(\sin x)^4 + \cdots$$

$$= 1 + \frac{1}{2}\left(x - \frac{x^3}{6} + \cdots\right) - \frac{1}{8}\left(x - \frac{x^3}{6} + \cdots\right)^2 + \frac{1}{16}(x - \cdots)^3 - \frac{5}{128}(x - \cdots)^4 + \cdots$$

$$= 1 + \frac{x}{2} - \frac{x^3}{12} - \frac{x^2}{8} + \frac{x^4}{24} + \frac{x^3}{16} - \frac{5x^4}{128} + \cdots$$

$$P_4(x) = 1 + \frac{x}{2} - \frac{x^2}{8} - \frac{x^3}{48} + \frac{x^4}{384}.$$

39. The series $\sum_{n=0}^{\infty} \frac{(-1)^n x^n}{(2n)!}$ is the Maclaurin series for $\cos x$ with x^2 replaced by x. For $x > 0$ the series therefore represents $\cos\sqrt{x}$. For $x < 0$, the series is $\sum_{n=0}^{\infty} \frac{|x|^n}{(2n)!}$, which is the Maclaurin series for $\cosh\sqrt{|x|}$. Thus the given series is the Maclaurin series for

$$f(x) = \begin{cases} \cos\sqrt{x} & \text{if } x \geq 0 \\ \cosh\sqrt{|x|} & \text{if } x < 0. \end{cases}$$

12. The Fourier cosine series of $f(t) = t$ on $[0, 1]$ has coefficients

$$\begin{aligned}
\frac{a_0}{2} &= \int_0^1 t\, dt = \frac{1}{2} \\
a_n &= 2\int_0^1 t\ \cos(n\pi t)\, dt \\
&= \frac{2(-1)^n - 2}{n^2\pi^2} = \begin{cases} 0 & \text{if } n \text{ is even} \\ \dfrac{-4}{n^2\pi^2} & \text{if } n \text{ is odd.} \end{cases}
\end{aligned}$$

The required Fourier cosine series is

$$\frac{1}{2} - \frac{4}{\pi^2}\sum_{n=1}^{\infty} \frac{\cos\big((2n-1)\pi t\big)}{(2n-1)^2}.$$