Math 421/510, Spring 2009, Midterm due on Monday March 2

Instructions

- The midtrerm will be collected at the end of lecture on Monday.
- Please do not discuss the questions among yourselves. But feel free to ask the instructor for hints and clarifications. The written solutions that you submit should be entirely your own.
- Answers should be clear, legible, and in complete English sentences. If you need to use results other than the ones discussed in class, state the result clearly with either a reference or a self-contained proof.

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1. In 1927, Schauder initiated the formal theory of bases in Banach spaces by offering up a basis for $C[0,1]$ that now bears his name. The purpose of this problem is to understand his construction.

   Consider the dyadic rationals in $[0,1]$, i.e., $\{r_{jk} = k2^{-j} : (j,k) \in \mathbb{Z}^2, j \geq 0, 0 \leq k \leq 2^j\}$. Enumerate these rationals according to the lexicographic order in (j,k) avoiding repetitions, so that

   $$t_0 = 0, \quad t_1 = 1, \quad t_2 = \frac{1}{2}, \quad t_3 = \frac{1}{4}, \quad t_4 = \frac{3}{4}, \quad \cdots .$$

   Let $f_0 \equiv 1$, $f_1(t) = t$. For $n \geq 2$, and $t_n = k_n 2^{-j_n}$ with $\gcd(k_n, 2) = 1$, define f_n to be the continuous, piecewise linear, tent-shaped function that vanishes outside $[t_n - 2^{-j_n}, t_n + 2^{-j_n}]$, and whose graph within this interval is given by the two lines joining the points $(t_n - 2^{-j_n}, 0)$ with $(t_n, 1)$ and $(t_n, 1)$ with $(t_n + 2^{-j_n}, 0)$ respectively. (Drawing a few pictures may help.)

   (a) Show that the set $\{f_n : n \geq 1\}$ is linearly independent. (*Hint* : Observe that $f_n(t_n) = 1$ and $f_k(t_n) = 0$ for $k > n$.)

   (b) Show that the span$\{f_0, \cdots, f_{2^m}\}$ is the set of all continuous piecewise linear or "polygonal" functions with nodes at the dyadic rationals $\{k2^{-m} : k = 0, 1, \cdots, 2^m\}$.

   (c) It remains to check that $\{f_n : n \geq 1\}$ is a Schauder basis for $C[0,1]$. How does one show that a countably infinite linearly independent set in a Banach space is a basic sequence? The following test for Schauder bases, due to Banach, is extremely useful:

   Theorem 1. *A sequence $\{\mathbf{x}_n : n \geq 1\}$ of nonzero vectors is a Schauder basis for the Banach space X if and only if*

(i) $\{\mathbf{x}_n : n \geq 1\}$ *has dense linear span in* X*, and*
(ii) there is a constant $K > 0$ *such that*

$$\left\|\sum_{i=1}^{n} a_i \mathbf{x}_i\right\| \leq K \left\|\sum_{i=1}^{m} a_i \mathbf{x}_i\right\|$$

for all scalars $\{a_i\}$ *and all* $n < m$*.*

We will soon be able to prove this result, but assuming it for now, show that $\{f_n\}$ is a Schauder basis for $C[0,1]$.

(d) In light of part (c), each $f \in C[0,1]$ can be uniquely written as a uniformly convergent series $f = \sum_{k=0}^{\infty} a_k f_k$. Describe the approximating polygonal functions, i.e., the partial sums of this expansion, in terms of f.

(e) It is tempting to wonder whether the monomials $\{t^n : n = 0, 1, 2, \cdots\}$ might form a Schauder basis for $C[0,1]$. Do they?

2. Next, let us apply ourselves to the task of finding a Schauder basis for $L^p[0,1]$, $1 \leq p < \infty$. The *Haar system* $\{h_n : n \geq 0\}$ on $[0,1]$ is defined by $h_0 \equiv 1$, and

$$h_{2^k+i}(x) = \begin{cases} 1 & \text{if } \frac{2i-2}{2^{k+1}} \leq x < \frac{2i-1}{2^{k+1}}, \\ -1 & \text{if } \frac{2i-1}{2^{k+1}} \leq x < \frac{2i}{2^{k+1}}, \\ 0 & \text{otherwise,} \end{cases}$$

for $k \geq 0$, and $1 \leq i \leq 2^k$. (Again, draw a few pictures.) Let $\mathcal{A}_k$ denote the collection of intervals

$$\mathcal{A}_k = \left\{ \left[\frac{i-1}{2^{k+1}}, \frac{i}{2^{k+1}}\right) : 1 \leq i \leq 2^{k+1} \right\}.$$

(a) Show that the linear span of $\{h_j : j \leq 2^{k+1}\}$ is the set of all step functions based on the intervals in $\mathcal{A}_k$, i.e.,

$$\text{span}\,\{h_0, \cdots, h_{2^{k+1}-1}\} = \text{span}\,\{\chi_I : I \in \mathcal{A}_k\}$$

Deduce from this that $\{h_n\}$ have dense linear span in $L^p[0,1]$.

(b) It remains to verify part (ii) of Banach's test. Show that this would follow if one can prove the inequality

$$|a+b|^p + |a-b|^p \geq 2|a|^p \quad \text{for all scalars } a \text{ and } b. \tag{1}$$

[*Hint* : Examine the supports of $\{h_n\}$, noting in particular that $\sum_{i=0}^{n} a_i h_i$ and $\sum_{i=0}^{n+1} a_i h_i$ differ only on the support of h_{n+1}.]

(c) Prove the inequality (1) by showing that $f(x) = |x|^p$ satisfies $f(x) + f(y) \geq 2f(\frac{x+y}{2})$ for all x, y.

3. (a) If $n \geq 1$, show that there is a measure μ on $[0,1]$ such that $p'(0) = \int p\, d\mu$ for every polynomial p of degree at most n.

(b) Does there exist a measure μ on $[0,1]$ such that $p'(0) = \int p\, d\mu$ for every polynomial p?

4. In class, we proved that the Fourier transform is an isometric isomorphism from $L^2[0,2\pi]$ onto $\ell^2(\mathbb{Z})$. An ingredient of the proof was the observation that the space of continuous functions on $[0,2\pi]$ is dense in $L^2[0,2\pi]$. In this problem, we investigate this issue in greater generality.
   (a) Let X be a locally compact Hausdorff space equipped with a Radon measure μ. Recall that $C_c(X)$ is the space of all $\mathbb{F}$-valued continuous functions on X with compact support. Show that $C_c(X)$ is dense in $L^p(X)$ for $1 \leq p < \infty$. [*Hint:* One method of proof uses the following measure-theoretic result, known as Lusin's theorem (look up the proof in Folland's Real Analysis or Rudin's Real and Complex Analysis, if you do not know it already):

   Theorem 2. *Let X be as above, A a measurable subset of X with $\mu(A) < \infty$, and suppose f is an $\mathbb{F}$-valued measurable function on X such that $f(x) = 0$ if $x \notin A$. Given any $\epsilon > 0$, there exists $g \in C_c(X)$ such that*
   $$\mu\left(\{x : f(x) \neq g(x)\}\right) < \epsilon.$$
   The function g may be chosen to further satisfy
   $$\sup_{x\in X} |g(x)| \leq \sup_{x\in X} |f(x)|.$$
   You may use this result without proof.]
   (b) If $X = \mathbb{R}^d$, $d \geq 1$, the result above may be strengthened as follows. Let $C_c^\infty(\mathbb{R}^d)$ denote the space of infinitely differentiable functions of compact support. Show that $C_c^\infty(\mathbb{R}^d)$ is dense in $L^p(\mathbb{R}^d)$, $1 \leq p < \infty$. Prove this.
   (c) The result that we needed for our proof (of the isometry of the Fourier transform) was that
   $$\mathcal{C} = \{f \in C[0,2\pi] : f(0) = f(2\pi)\}$$
   is dense in $L^2[0,2\pi]$. Explain why this follows from the results above.
   (d) Do these approximation theorems hold for $p = \infty$?