

Math 421/510, Spring 2009, Homework Set 2

due on Monday March 23

Instructions

- Homework will be collected at the end of lecture on Friday.
- You are encouraged to discuss homework problems among yourselves. Also feel free to ask the instructor for hints and clarifications. However the written solutions that you submit should be entirely your own.
- Answers should be clear, legible, and in complete English sentences. If you need to use results other than the ones discussed in class, state the result clearly with either a reference or a self-contained proof.

This homework set will cover different aspects of the general Banach space results (such as Hahn-Banach, open mapping, closed graph, uniform boundedness theorems) that we have been discussing in class and their applications.

1. Let \mathbb{H} be a separable Hilbert space with an orthonormal basis $\{\mathbf{e}_n : n \geq 1\}$, and $\mathbb{M} \subseteq \mathbb{N} = \{1, 2, 3, \dots\}$. We proved a result in class that gives a formula for the orthogonal projection operator of \mathbb{H} onto the closed linear span of $\{\mathbf{e}_m : m \in \mathbb{M}\}$. We also know that these projection operators are continuous, i.e., bounded linear maps. Can this notion of projection onto coordinate subspaces be generalized to a Banach space with a Schauder basis (which now provides the analogue of an orthonormal basis)? The current problem explores this question.

Given a Schauder basis $\{\mathbf{x}_n : n \geq 1\}$ in a Banach space X , let us define the *coordinate functionals* $x_n^* : X \rightarrow \mathbb{R}$ by

$$x_n^*(\mathbf{x}) = a_n \quad \text{where} \quad \mathbf{x} = \sum_{i=1}^{\infty} a_i \mathbf{x}_i.$$

We also define a sequence of linear maps $\{P_n : n \geq 1\}$ on X by

$$P_n(\mathbf{x}) = \sum_{i=1}^n x_i^*(\mathbf{x}) \mathbf{x}_i.$$

Thus P_n may be viewed as a projection onto the linear span of $\{\mathbf{x}_i : 1 \leq i \leq n\}$. Obviously the maps $\{x_n^*\}$ are all continuous precisely when the P_n -s are all continuous (why?). The amazing fact

that all these operations are continuous whenever X is complete is due to Banach.

- (a) Prove that if $\{\mathbf{x}_n : n \geq 1\}$ is a Schauder basis for a Banach space X , then every P_n (and hence every x_n^*) is continuous. Moreover $K = \sup_n \|P_n\| < \infty$. (*Hint*: Banach's ingenious idea was to define a new norm on X by setting

$$\|\mathbf{x}\| = \sup_n \|P_n \mathbf{x}\|.$$

Check that this is well-defined and a norm, and use it to prove the result stated above.)

- (b) We are now in a position to prove the “test for Schauder basis” that we used a few times in an earlier problem set. Recall the result and prove it.
2. (a) If K is the closure of the unit ball in the complex plane and $a \in K$, find a complex regular Borel measure μ supported on the boundary of K such that

$$(1) \quad f(a) = \int f d\mu$$

for all functions f that are analytic in the open disk and continuous up to the boundary.

- (b) Is the measure that you found in part (a) a probability measure? (A probability measure is a non-negative measure μ such that $\|\mu\| = 1$.) If not, find a probability measure μ supported on ∂K with the same property (1). (*Hint*: A quick review of harmonic extensions might be useful.) In particular, this shows that the choice of μ is not unique.
- (c) Let's now try to generalize the two observations above to other planar domains. Let K be a compact subset of \mathbb{C} with nonempty interior and smooth boundary and define

$$A(K) = \{f \in C(K) : f \text{ is analytic on the interior of } K\}.$$

Here $C(K)$ is the normed space (over \mathbb{C}) of continuous (complex-valued) functions on K . Show that if $a \in K$, then there is a complex regular Borel measure μ supported on ∂K such that

$$f(a) = \int f d\mu$$

for every $f \in A(K)$. Do this in two ways, once by appealing to Hahn-Banach, and also from standard results in complex function theory.

- (d) Show next that it is possible to choose the measure μ in part (c) to be a probability measure.

3. If K is a compact subset of \mathbb{R} , Stone-Weierstrass theorem tells us that the polynomials are dense in $C(K)$. What if we consider the space $C_0(\mathbb{R})$, i.e., the space of all continuous functions on \mathbb{R} that vanish at infinity (equipped with the sup norm of course), and wish to find a subset of functions of known functional form that has dense linear span in $C_0(\mathbb{R})$? The following result, which you should prove, furnishes an answer.

Let $w(t)$ be a given positive function on \mathbb{R} that decays exponentially as $|t| \rightarrow \infty$,

$$0 < w(t) < ae^{-c|t|}, \quad c > 0.$$

Show that the functions $\{t^n w(t) : n \geq 1\}$ belong to $C_0(\mathbb{R})$ and their closed linear span is all of $C_0(\mathbb{R})$.

4. A very important geometric application of the Hahn-Banach theorem is to separation results concerning convex sets. Here is an example. Follow the steps outlined below to prove two versions of the hyperplane separation theorem.

- (a) Let X be a linear space over the reals, K a convex set that has an interior point, which we take to be the origin. Denote the gauge p_K with respect to the origin as follows:

$$p_K(x) = \inf \left\{ a : a > 0, \frac{x}{a} \in K \right\}.$$

Show that p_K is well-defined, positive homogeneous and subadditive.

- (b) Show that for any convex set K as in part (a),

$$p_K(x) \leq 1 \text{ if } x \in K,$$

$$p_K(x) < 1 \text{ if and only if } x \text{ is an interior point of } K.$$

- (c) We turn now to the notion of *hyperplane*. Suppose that ℓ is a linear functional not $\equiv 0$. The set $\{x : \ell(x) = c\}$ is called a hyperplane. The sets $\{x : \ell(x) < c\}$ and $\{x : \ell(x) > c\}$ are called open *halfspaces*. The sets $\{x : \ell(x) \geq c\}$ and $\{x : \ell(x) \leq c\}$ are called closed halfspaces. Prove:

Theorem 1 (Hyperplane Separation Theorem). *Let K be a nonempty convex subset of a linear space X over the reals. Suppose that all points of K are interior. Any point y not in K can be separated from ℓ by a hyperplane $\ell(x) = c$; that is there is a linear functional ℓ depending on y such that*

$$\ell(x) < c \quad \text{for all } x \in K, \quad \ell(y) = c.$$

Theorem 2 (Extended Hyperplane Separation). *Let X be a linear space over \mathbb{R} ; H and M disjoint, convex subsets of X , at least one of which has an interior point. Then H and M can be separated by a hyperplane $\ell(x) = c$; that is, there is a nonzero linear functional ℓ and a number c such that*

$$\ell(u) \leq c \leq \ell(v)$$

for all $u \in H$, $v \in M$.

5. (a) We have seen plenty of examples where we have used properties of a Banach space X to deduce structural properties of its dual space X^* . Can one use topological information of X^* to deduce similar properties of X ? Here is a result along these lines. Prove it.

Let Z be a normed space over \mathbb{C} . If Z^* is separable, so is Z .

- (b) We may have briefly touched upon the notion of reflexivity in class. Recall that a Banach space is called reflexive if $X = X^{**}$. (Thus the spaces L^p , $1 < p < \infty$ are all reflexive.) Use the result in part (a) to prove that $C[0, 1]$ is not reflexive.
- (c) While we are on the topic of reflexive spaces, here is a result that is of some interest, and one that we will need to use soon. Prove it.

A closed linear subspace Y of a reflexive Banach space X is reflexive.

6. Have you wondered about extending a general linear operator on a normed linear space, instead of a linear functional? In other words, is there an analogue of the Hahn-Banach theorem if (instead of the scalar field) the range space of the operator to be extended is an arbitrary Banach space? In this problem, we discuss a quantitative notion connected with extensions of linear operators, called *extension constant*.

We say that a Banach space X is *injective* if for every Banach space Y , every subspace Z of Y and every operator $T : Z \rightarrow X$ there exists an extension $\tilde{T} : Y \rightarrow X$. We define the *extension constant* $e(X)$ by

$$e(X) = \inf \left\{ c \mid \begin{array}{l} \text{for every } Y \supset Z \text{ and } T : Z \rightarrow X \text{ there exists} \\ \tilde{T} : Y \rightarrow X \text{ such that } \tilde{T} = T \text{ on } Z, \text{ and } \|\tilde{T}\| \leq c\|T\| \end{array} \right\}$$

Think about what $e(X)$ measures, and convince yourself that $X = \mathbb{F}$ is injective.

- (a) Show that every injective space has $e(X) < \infty$.
- (b) Show that $e(\ell^\infty) = 1$.