Math 421/510, Spring 2009, Homework Set 2
due on Friday February 13

Instructions

- Homework will be collected at the end of lecture on Friday.
- You are encouraged to discuss homework problems among yourselves. Also feel free to ask the instructor for hints and clarifications. However the written solutions that you submit should be entirely your own.
- Answers should be clear, legible, and in complete English sentences. If you need to use results other than the ones discussed in class, state the result clearly with either a reference or a self-contained proof.

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1. Let X be an uncountable set, $\mathcal{P}(X)$ the power set of X, and ν the counting measure on $(X, \mathcal{P}(X))$. Let Ω be the σ-algebra of countable or co-countable sets, i.e., $A \in \Omega$ if and only if either A or A^c is countable. Let μ be the restriction of ν to Ω.
   (a) Show that $L^1(\nu) = L^1(\mu)$.
   (b) Describe $L^\infty(\nu)$ and $L^\infty(\mu)$. Show that $L^\infty(\mu) \subsetneq L^\infty(\nu)$.
   (c) Describe $L^1(\mu)^*$. Use this to deduce that the mapping from $L^\infty(X, \Omega, \mu)$ to $(L^1(X, \Omega, \mu))^*$ that sends $g \mapsto F_g$, where

$$F_g(f) = \int_X fg d\mu, \quad f \in L^1(\mu) \tag{1}$$

   is in general not surjective.
2. Let (X, Ω, μ) be an arbitrary measure space, and recall the definition of F_g from (1). We have shown in class that the mapping $\Phi : L^\infty(X, \Omega, \mu) \to (L^1(X, \Omega, \mu))^*$ which sends $g \mapsto F_g$ is in general not injective. The example in the preceding problem shows that in general Φ need not be surjective either. However, if (X, Ω, μ) is a σ-finite measure space, then Φ is indeed an isometric isomorphism between L^∞ and $(L^1)^*$. Prove this.
3. Let $X = C[0, 1]$ equipped with the $L^1([0, 1], dx)$ norm. Define $L : X \to \mathbb{F}$ by $L(f) = f(\frac{1}{2})$. Prove that L is an unbounded linear functional in two ways, once by choosing a sequence $\{f_n\} \subseteq X$ of functions with unit norm for which $|L(f_n)| \to \infty$, and also by appealing to the result in problem 2.

4. For $n \geq 1$, let $C^{(n)}[0,1]$ denote the space of continuous functions f on $[0,1]$ that are n times continuously differentiable. Convince yourself, but do not submit a proof, that $C^{(n)}[0,1]$ equipped with the norm
$$||f|| = \max_{0\leq k\leq n} \sup_{x\in[0,1]} |f^{(k)}(x)|$$
is a Banach space. The goal of this problem is to identify the dual of this space.
(a) Show that
$$|||f||| = \sum_{k=0}^{n-1} \left|f^{(k)}(0)\right| + \sup_{x\in[0,1]} |f^{(n)}(x)|$$
is an equivalent norm on $C^{(n)}[0,1]$.
(b) Show that $L \in (C^{(n)}[0,1])^*$ if and only if there are scalars $\{\alpha_k : 0 \leq k \leq n-1\}$ and a complex regular Borel measure μ on $[0,1]$ such that
$$L(f) = \sum_{k=0}^{n-1} \alpha_k f^{(k)}(0) + \int f^{(n)}\, d\mu.$$
(c) Determine $||L||$ in terms of $\{|\alpha_k| : 0 \leq k \leq n-1\}$ and the total variation norm of μ.

5. Let $\mathbb{H}$ be the Hilbert space of all absolutely continuous functions $f : [0,1] \to \mathbb{F}$ such that $f(0) = 0$ and $f' \in L^2[0,1]$. The inner product on $\mathbb{H}$ is
$$\langle f, g\rangle = \int f'(t)\overline{g'(t)}\, dt.$$
(a) Given a fixed $t \in (0,1]$, show that the evaluation map L_t at t given by
$$L_t(h) = h(t)$$
is a bounded linear functional on $\mathbb{H}$.
(b) By part (a) and according to the Riesz representation theorem for Hilbert spaces, there exists a unique vector $h_t \in \mathbb{H}$ such that $L_t(\cdot) = \langle \cdot, h_t\rangle$. Find h_t.
(c) Determine $||L_t||$.