Math 421/510, Spring 2008, Midterm
(Due date: Wednesday March 12)

Instructions

- The midterm will be collected at the end of lecture on Wednesday. **There will be no extensions for the midterm.**
- Unlike homework assignments, you must work on the midterm on your own. If you need hints or clarifications, please feel free to talk to the instructor.
- Answers should be clear, legible, and in complete English sentences. Solutions must be self-contained – only results proved in class can be used without proof.

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1. The closed span of an orthonormal system in a Hilbert space remains stable under small perturbations, in the following sense. Let $\{\varphi_n : n \geq 1\}$ be an orthonormal basis of a Hilbert space $\mathbb{H}$. If $\{\psi_n : n \geq 1\}$ is an orthonormal system in $\mathbb{H}$ such that
$$\sum_n ||\varphi_n - \psi_n||^2 < \infty,$$
then $\{\psi_n : n \geq 1\}$ is an orthonormal basis for $\mathbb{H}$. Prove this.
2. Let $(X, || \cdot ||)$ be a normed linear space over $\mathbb{R}$ such that
$$||\mathbf{x} + \mathbf{y}|| < ||\mathbf{x}|| + ||\mathbf{y}|| \quad \text{if } \mathbf{x} \neq \alpha \mathbf{y} \text{ for any } \alpha \geq 0.$$
Such a norm is said to be *strictly subadditive.* Show that any isometry of X into itself that fixes the origin is linear. Use this to characterize all isometries of X into itself.

   Remark: This is a special case of the following more general result due to Mazur and Ulam. Let X and X' be two normed linear spaces over the reals , φ an isometric mapping of X onto X' that carries $\mathbf{0}$ to $\mathbf{0}$. Then φ is linear.
3. Identify all linear isometries of a Hilbert space $\mathbb{H}$.
4. Consider the sequence space $c_0(\mathbb{N})$.
   (i) Let $p : \mathbb{N} \to \mathbb{N}$ be a permutation (i.e. a bijection) of the positive integers. Check that the map
   $$\{a_n : n \geq 1\} \mapsto \{a_{p(n)} : n \geq 1\}$$
   is an isometry on c_0.
   (ii) Let $\{b_n : n \geq 1\}$ be an arbitrary sequence of complex numbers such that $|b_n| = 1$ for all n. Check that
   $$\{a_n : n \geq 1\} \mapsto \{b_n a_n : n \geq 1\}$$
   is an isometry on c_0.
   (iii) Show that every linear isometry of c_0 onto itself is a composition of the types described in (i) and (ii).