Math 421/510, Spring 2007, Homework Set 4
(Suggested due date: Friday March 30)

Instructions

- Homework will be collected at the end of lecture on Friday.
- You are encouraged to discuss homework problems among yourselves. Also feel free to ask the instructor for hints and clarifications. However the written solutions that you submit should be entirely your own.
- Answers should be clear, legible, and in complete English sentences. If you need to use results other than the ones discussed in class, provide self-contained proofs.

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1. Recall that
   (i) a subset M of a topological space X is said to be *nowhere dense in* X if the closure of M does not contain any nonempty open set of X, and
   (ii) $N \subset X$ is *dense* if the closure of N equals X.
   (iii) M is said to be of the *first category* if M is expressible as the union of a countable number of sets each of which is nowhere dense in X. Otherwise M is said to be of the *second category*.

   Use the definitions above to answer the following questions.
   (a) Given a sequence of bounded linear operators $\{T_n\}$ defined on a Banach space X, with $T_n \in \mathcal{B}(X, Y_n)$ where Y_n is a normed linear space, the set
   $$B = \left\{ x \in X : \limsup_{n\to\infty} ||T_n x|| < \infty \right\}$$
   either coincides with X or is of the first category.
   (b) Let $\{T_{p,q} : q \geq 1\}$ be a sequence of bounded linear operators defined on a Banach space X into a normed linear space Y_p ($p = 1, 2, \cdots$). Suppose that for each p, there exists $x_p \in X$ such that $\limsup_{q\to\infty} ||T_{p,q} x_p|| = \infty$. Then the set
   $$B = \left\{ x \in X : \limsup_{q\to\infty} ||T_{p,q} x|| = \infty \text{ for all } p = 1, 2, \cdots \right\}$$
   is of second category. This result is often referred to as the *principle of condensation of singularities* (make sure you understand why).
2. Consider the complex Banach space
   $$\mathfrak{X} = \{f \in C([0, 2\pi]) : f(0) = f(2\pi)\}$$
   with sup norm.

(a) For each n, let T_n be the operator in $\mathcal{B}(\mathfrak{X})$ that assigns to f the nth partial sum of its Fourier series, i.e.

$$T_n f = \sum_{k=-n}^{n} \langle f, e_k \rangle e_k$$

where $e_k(x) = e^{ikx}$. Show that $||T_n|| \to \infty$ as $n \to \infty$ and deduce the existence of a function f whose Fourier series is not uniformly convergent.
Hint : Write T_n as an integral operator with kernel D_n, namely

$$T_n f(x) = \int D_n(x-y) f(y) dy.$$

Obtain a closed form expression for D_n. Show that $||T_n|| = ||D_n||_1$, and that the latter grows like $\log n$. The function D_n is called the *Dirichlet kernel*.

(b) Use the principle of condensation of singularities to show that there exists a function $f \in \mathfrak{X}$ such that $\limsup_n |T_n f| = \infty$ on an uncountable set.

3. (a) Prove the Hellinger-Toeplitz theorem stated in class.
   (b) Extend the theorem to include pairs of operators A, B satisfying $\langle Ax, y \rangle = \langle x, By \rangle$.
4. (Reading Exercise, Conway V.5) Consider the space ℓ^1 both as a Banach space and as the locally convex topological space with the weak topology induced by its dual space ℓ^∞. Show that every weakly convergent sequence in ℓ^1 is norm convergent. Explain why this fact does not imply that the two topologies coincide.
5. Let X be a Banach space in either of the norms $|| \cdot ||^{(1)}$ and $|| \cdot ||^{(2)}$. Suppose that $|| \cdot ||^{(1)} \leq C || \cdot ||^{(2)}$ for some C. Show that the topologies induced by the two norms are equivalent.
6. Recall that the convex hull of a set A in a TVS $\mathfrak{X}$ is defined to be the smallest convex set in $\mathfrak{X}$ containing A.
   (a) Show that in an LCS, the convex hull of every bounded set is bounded.
   (b) For $0 < p < 1$ let $\ell^p =$ all sequences x such that $\sum_{n=1}^{\infty} |x(n)|^p < \infty$. Define

$$d(x, y) = \sum_{n=1}^{\infty} |x(n) - y(n)|^p \quad \text{(no } p\text{th root).}$$

   Show that d is a metric, but the metric space (ℓ^p, d) is not locally convex.
7. (More on the space of test functions $\mathcal{D}(\Omega)$) Recall that a subset B in a linear topological space X is said to be *bounded* if it is *absorbed* by any neighborhood U of 0, i.e., if there exists a positive constant α such that $\alpha B \subseteq U$. A locally convex space X is called *bornologic* if it satisfies the condition:

   If a convex balanced set M of X absorbs every bounded set of X,
   then M is a neighborhood of 0 at X.

   (a) Prove that a locally convex space is bornologic if and only if every seminorm on X which is bounded on every bounded set is continuous.

(b) Show that the space $\mathcal{D}(\Omega)$ is bornologic. [*Hint : A linear operator T on one locally convex space into another is continuous if and only if T maps bounded sets into bounded sets.*]