Math 421/510, Spring 2007, Homework Set 2
(Suggested due date: Friday February 9)

Instructions

- Homework will be collected at the end of lecture on Wednesday.
- You are encouraged to discuss homework problems among yourselves. Also feel free to ask the instructor for hints and clarifications. However the written solutions that you submit should be entirely your own.
- Answers should be clear, legible, and in complete English sentences. If you need to use results other than the ones discussed in class, provide self-contained proofs.

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1. Let $\{\varphi_1, \cdots, \varphi_n\}$ be an orthonormal set in a separable Hilbert space $\mathcal{H}$, and let
$$\{\psi_1, \cdots, \psi_n\} \subseteq [\text{span}\{\varphi_1, \cdots, \varphi_n\}]^\perp$$
For $h \in \mathcal{H}$, define the finite rank operator K as follows,
$$Kh = \sum_{i=1}^{n} \langle h, \varphi_i \rangle \psi_i.$$
Prove that $I + \alpha K$ is invertible for any α and find its inverse.
2. (a) (The Schur test) Given a doubly infinite matrix $A = ((\alpha_{nm}))$ and a sequence $\{a_n\} \subset \mathbb{R}_+$ such that
$$\sum_{n=1}^{\infty} |\alpha_{nm}| a_n \leq b a_m \qquad \text{for every } m$$
$$\sum_{m=1}^{\infty} |\alpha_{nm}| a_m \leq c a_n \qquad \text{for every } n$$
for suitable constants b and c. Show that there is $T \in \mathcal{B}(\mathcal{H})$ with A as its matrix, and that $||T||^2 \leq bc$.
(b) (The Hilbert matrix) Show that there exists $T \in \mathcal{B}(\mathcal{H})$ whose matrix relative to an orthonormal basis $\{e_n : n \in \mathbb{N}\}$ is given by $A = ((\alpha_{nm}))$, with $\alpha_{nm} = (n + m - 1)^{-1}$. Show that $T = T^*$ and that $||T|| \leq \pi$.
Hint: Use the Schur test with $a_n = (n - \frac{1}{2})^{-1/2}$.
3. Recall the multiplication operator M_ϕ on $L^2[a, b]$ that we discussed in class, $\phi \in L^\infty[a, b]$.
(a) Can M_ϕ be a compact operator for $\phi \not\equiv 0$?
(b) Describe the spaces $\ker(M_\phi)$ and $\overline{\text{Ran}(M_\phi)}$, where $\phi \in L^\infty[a, b]$ and M_ϕ is the multiplication operator on $L^2[a, b]$ that we discussed in class.
(c) Suppose that ϕ is a polynomial. Find a necessary and sufficient condition on ϕ for $\text{Ran}(M_\phi)$ to be closed.

4. In this problem, we continue our study of the unilateral shift operators, namely the right shift S_r and the left shift S_ℓ on $\ell^2(\mathbb{N})$.
   (a) For S_r, find S_r^*.
   (b) Show that S_r has no eigenvalues, but that every $\lambda \in \mathbb{C}$ with $|\lambda| < 1$ is an eigenvalue for S_r^* with multiplicity 1.
   (c) Show that none of the eigenvalues of S_r^* are orthogonal to each other.
   (d) Prove that for $|\mu| < 1$, both $I - \mu S_r$ and $I - \mu S_\ell$ are invertible. What happens when $|\mu| \geq 1$?
   (e) For $|\mu| > 1$, describe $\ker(I - \mu S_\ell)$ and $\operatorname{Ran}(I - \mu S_r)^\perp$, and find their dimensions.
5. In this problem, we explore another application of functional-analytic techniques to solving integral equations. Given $k \in L^2([a,b] \times [a,b])$ and $g \in L^2[a,b]$, consider the integral equation

$$f(t) - \int_a^b k(t,s) f(s) ds = g(t). \tag{1}$$

   Here "=" means equal almost everywhere.
   (a) Show that if $||k||_{L^2} < 1$, there exists a unique solution $f \in L^2[a,b]$ of the above integral equation.
   (b) (*The earlier version of this problem was poorly worded. Sorry about the confusion.*) If in addition

$$\sup_t \int_a^b |k(t,s)|^2 ds = c < \infty, \tag{2}$$

   find an explicit expression for the solution f of (1) in terms of the given data g. In particular, identify f as an integral operator acting on g, and estimate its operator norm in terms of $||k||_2$ and the constant c in (2).
   (c) Find an explicit solution of the integral equation

$$f(t) - \lambda \int_0^1 e^{\lambda(t-s)} f(s)\, ds = g(t) \in L^2[0,1].$$

6. Show that the collection of all finite-rank operators on a Hilbert space $\mathcal{H}$ is a minimal ideal of $\mathcal{B}(\mathcal{H})$ (i.e., it is an ideal that does not contain any nontrivial ideals).