

Math 421/510, Spring 2007, Homework Set 1

(due on Wednesday January 24)

Instructions

- Homework will be collected at the end of lecture on Wednesday.
- You are encouraged to discuss homework problems among yourselves. Also feel free to ask the instructor for hints and clarifications. However the written solutions that you submit should be entirely your own.
- Answers should be clear, legible, and in complete English sentences. If you need to use results other than the ones discussed in class, provide self-contained proofs.

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1. Let $\mathcal{E} = \{e_n : n \in \mathbb{N}\}$ be an orthonormal basis for a Hilbert space \mathcal{H} . If $\mathcal{F} = \{f_n : n \in \mathbb{N}\}$ is an orthonormal set such that

$$(1) \quad \sum_{n=1}^{\infty} \|e_n - f_n\|^2 < \infty,$$

then show that \mathcal{F} is an orthonormal basis for \mathcal{H} . Thus an orthonormal system that is close enough (in the sense of (1)) to an orthonormal basis is also an orthonormal basis.

2. If $\{h \in \mathcal{H} : \|h\| \leq 1\}$ is compact, show that $\dim \mathcal{H} < \infty$. In other words, the equivalence “closed and bounded \iff compact” is not true in an infinite-dimensional Hilbert space!
3. If \mathcal{H} is an infinite-dimensional Hilbert space, show that no orthonormal basis for \mathcal{H} is a Hamel basis. Show that a Hamel basis is uncountable.
4. Find an example of a nonseparable Hilbert space.
5. Let Ω be a domain in \mathbb{C} , and let

$$\mathcal{H} := L^2(\Omega, \mathbb{C}) = \left\{ f : \Omega \rightarrow \mathbb{C} \mid \iint_{\Omega} |f(x + iy)|^2 dx dy < +\infty \right\}.$$

Let \mathcal{X} denote a closed subspace of \mathcal{H} . A complex-valued function $K(\cdot, \cdot) : \Omega \times \Omega \rightarrow \mathbb{C}$ is called a *reproducing kernel* of \mathcal{X} if it satisfies the conditions

- (i) for every fixed w , $K(\cdot, w) \in \mathcal{X}$, and
- (ii) $f(z) = \iint_{\Omega} K(z, w) f(w) d\mu(w)$,

where $w = x + iy$ and $d\mu(w) = dx dy$.

- (a) Prove the following result on the existence of reproducing kernels: \mathcal{X} has a reproducing kernel K if and only if there exists, for any $y \in \Omega$, a positive constant C_y such that

$$|f(y)| \leq C_y \|f\| \quad \text{for all } f \in \mathcal{X}.$$

- (b) Assuming that \mathcal{X} is separable and has a reproducing kernel K , find an explicit expression for K in terms of the orthonormal basis for \mathcal{X} .
- (c) Show that $\mathcal{X} = A^2(\Omega)$ (the subspace of \mathcal{H} consisting of complex-analytic square-integrable functions on Ω) has a reproducing kernel (called the *Bergman kernel*).
- (d) Find a domain $\Omega \subsetneq \mathbb{C}$ for which you can explicitly compute the Bergman kernel.
6. Let $L_0 = \{\varphi \in L^2[-a, a] : \varphi(t) = -\varphi(-t) \text{ a. e.}\}$.
- (a) Find L_0^\perp .
- (b) Show that both L_0 and L_0^\perp are closed infinite dimensional subspaces of $L^2[-a, a]$.
- (c) For $f \in L^2[-a, a]$, find its projection onto L_0 and L_0^\perp .
- (d) Find the distances of L_0 and L_0^\perp from $f(t) = t^2 + t$.
7. Let $\mathcal{H} = L^2(0, 1)$ and let C^1 be the set of all continuous functions on $[0, 1]$ that have a continuous derivative. Let $t \in [0, 1]$ and define $L : C^1 \rightarrow \mathbb{F}$ by $L(h) = h'(t)$. Show that there is no bounded linear functional on \mathcal{H} that agrees with L on C^1 .