Math 320, Fall 2018
Midterm 1 Solution

Name: SID:

Instructions

- The total time is 50 minutes.
- The maximum score is 100 points.
- Use the reverse side of each page if you need extra space.
- Show all your work. A correct answer without intermediate steps will receive no credit.
- Partial credit will be assigned to the clarity and presentation style of solutions. Please ensure that your answers are effectively comunicated.
- No clarification will be given for any problems; if you believe a problem is ambiguous, interpret it as best you can and write down any assumptions you feel are necessary.

Problem	**Points**	**Score**
1	$10 \times 3 = 30$	
2	20	
3	20	
4	30 + 10 (extra credit)	
MAX	100	

1. For each term below, give a complete definition and an example. Prove (or demonstrate) that your example matches the definition that you give.

   (a) Supremum of a set $A \subseteq \mathbb{R}$.

   (5 + 5 = 10 points)

   Solution. Let A be a nonempty subset of $\mathbb{R}$ that is bounded above. A point $a_0 \in \mathbb{R}$ is said to be the *supremum of* A, *denoted sup*(A) if both of the following conditions hold:

   - $a \leq a_0$ for all $a \in A$
   - if $b \in \mathbb{R}$ is an upper bound of A (i.e., $a \leq b$ for all $a \in A$), then $a_0 \leq b$.

   Example: Let $A = [0, 1]$, then $\sup(A) = 1$. □

   (b) A limit point of a subset E in a metric space (X, d).

   (5+5 = 10 points)

   Solution. A point $p \in X$ is said to be a *limit point* of $E \subset X$ if for every $\epsilon > 0$, there exists $q \neq p$, $q \in E$ such that $d(p, q) < \epsilon$.

   Example: The origin 0 is a limit point of $E = \{1/n : n \geq 1\} \subseteq \mathbb{R}$. □

(c) A countably infinite set.

(5+5 = 10 points)

Solution. A set A is called *countably infinite* if there exists a bijection $f : \mathbb{N} \to A$, where $\mathbb{N}$ denotes the set of natural numbers.

Example: The set of positive even integers $A = \{2, 4, \cdots\}$ is countably infinite; $f(n) = 2n$ provides the bijection from $\mathbb{N}$ onto A. $\square$

2. A number $\alpha \in \mathbb{R}$ is called *algebraic* if there exists a non-zero polynomial P with integer coefficients so that $P(\alpha) = 0$. A real number that is not algebraic is called *transcendental*. Prove that the set of transcendental numbers is uncountable.

(20 points)

Solution. First, observe that for each $n \in \mathbb{N}$, the set of polynomials of degree $\leq n$ with integer coefficients is countable, since it can be put in bijective correspondence with $\mathbb{Z}^{n+1}$ via the bijection $(a_0, ...a_n) \mapsto P(x) = a_n x^n + \ldots + a_0$. Thus the set of polynomials with integer coefficients is a countable union of countable sets, and is thus countable. For each polynomial P, let $S_P = \{x \in \mathbb{R} \colon P(x) = 0\}$. This set is finite (indeed, it has cardinality at most the degree of P). Thus $A = \bigcup_P S_P$ is a countable union of countable sets, and is thus countable, where the union is taken over all non-zero polynomials with integer coefficients. However, the set A is precisely the set of algebraic numbers. We conclude that A is countable.

Suppose if possible that the set A^c of transcendental numbers is countable. This would imply that $\mathbb{R} = A \cup A^c$ is the union of two countable sets, hence countable. But this contradicts the fact (proved in class and the textbook) that $\mathbb{R}$ is uncountable. $\square$

3. Define $C_0 = [0,1]$; this is a union of $2^0 = 1$ closed intervals, each of length $3^0 = 1$. Define $C_1 = [0, \frac{1}{3}] \cup [\frac{2}{3}, 1]$; this set contains 2^1 intervals, each of length 3^{-1}; it is obtained by removing the middle third of each interval from C_0. For each $i = 2, 3, \dots,$ define C_i to be the union of 2^i closed intervals, each of length 3^{-i}, obtained by removing the middle third of each of the intervals from C_{i-1}. Define $\mathcal{C} = \bigcap_{i=0}^{\infty} C_i$.

   Prove that $\mathcal{C}$ does not contain a non-empty open interval.

(20 points)

Solution. We will show that given any $x, y \in \mathcal{C}$, $x < y$,

$$\text{there exists } z \in [0,1] \setminus \mathcal{C} \text{ such that } x < z < y. \tag{1}$$

We have been given that

$$\mathcal{C} = \bigcap_{n=1}^{\infty} C_n,$$

where C_n, the set obtained at the n-th step of the Cantor construction, is a disjoint union of 2^n closed intervals (called n-th stage *basic intervals*), each of length 3^{-n}. In particular, the length of an nth stage basic interval goes to zero as $n \to \infty$. Therefore, given $x, y \in \mathcal{C}$, $x < y$, one can find $m_0 \geq 1$ such that $3^{-m} < |x - y|$ for all $m \geq m_0$. This means that x and y cannot lie in the same m-th basic interval for any $m \geq m_0$. Let n denote the largest positive integer such that both x and y lie inside a common n-th stage basic interval, say $I = [a, b]$. Since n has to be a non-negative integer smaller than m_0, such an integer must exist.

At the $(n+1)$-th step, I is decomposed into three equal and disjoint pieces

$$I = \bigcup_{j=1}^{3} I_j, \quad \text{with } I_1 = \left[a, a + \frac{b-a}{3}\right], \quad I_3 = \left[b - \frac{b-a}{3}, b\right]$$

and the middle third portion I_2 is thrown away. In particular, $z = a + (b - a)/2 = (a + b)/2 \notin \mathcal{C}$. By the maximality of n, we also know that $x \in I_1$ and $y \in I_3$, proving (1). $\square$

4. Give brief answers to the questions below. Your answer should be in the form of a short proof or a counterexample, as appropriate.

   (a) Let d be the usual metric on $\mathbb{R}$, i.e. $d(x, y) = |x - y|$. If $E = \mathbb{Q} \cap [0, 1] \subset \mathbb{R}$, what is its closure $\overline{E}$?

   (10 points)

   Solution. $\overline{E} = [0, 1]$. This is because rationals are dense; for any $x \in [0, 1]$ and any $\epsilon > 0$, there exists $r \in E \cap (x - \epsilon, x + \epsilon)$. □

(b) Determine whether the following statement is true or false. For sets $A, B \subseteq \mathbb{R}$ that are bounded above, one always has

$$\sup(A - B) = \sup(A) - \sup(B).$$

Here $A - B = \{a - b : a \in A, b \in B\}$.

(10 points)

Solution. The statement is false. Choose $A = B = [-1, 1]$. Then $A - B = [-2, 2]$, so $\sup(A - B) = 2$, whereas $\sup(A) - \sup(B) = 0$. $\square$

(c) Let d_1 and d_2 be two metrics on $\mathbb{R}^2$ given by

$$d_1(x, y) = |x_1 - y_1| + |x_2 - y_2|,$$

$$d_2(x, y) = \left[(x_1 - y_1)^2 + (x_2 - y_2)^2\right]^{\frac{1}{2}}$$

for all points $x = (x_1, x_2),\ y = (y_1, y_2) \in \mathbb{R}^2$. Show that

$$\frac{1}{\sqrt{2}} d_1(x, y) \leq d_2(x, y) \leq d_1(x, y) \text{ for all } x, y \in \mathbb{R}^2.$$

(10 points)

Proof. Given any two non-negative reals a and b, it is easy to see that $a^2 + b^2 \leq a^2 + b^2 + 2ab = (a+b)^2$. It also follows from the Cauchy-Schwarz inequality that $(a + b) \leq \sqrt{2}\sqrt{a^2 + b^2}$. Setting $a = |x_1 - y_1|$, $b = |x_2 - y_2|$ leads to the desired inequalities. □

(d) (Extra credit) Determine whether the following statement is true or false. There exists an uncountable collection of sets $\{\mathbb{S}_\alpha\}$ such that

- Each set $\mathbb{S}_\alpha$ is a subset of $\mathbb{N}$
- each $\mathbb{S}_\alpha$ is infinite
- For every pair $\mathbb{S}_\alpha$ and $\mathbb{S}_\beta$ with $\alpha \neq \beta$, we have that $\mathbb{S}_\alpha \cap \mathbb{S}_\beta$ is finite

Here $\mathbb{N}$ denotes the set of positive integers.

(10 points)

Solution. Let A denote all infinite binary sequences (whose entries are either 0 or 1) consisting of infinitely many 1-s. We know that A is uncountable. Let $\mathcal{P} = \{p_1 < p_2 < p_3 < \cdots\}$ be the set of primes. For each $\overline{\alpha} = (\alpha_1, \alpha_2, \cdots) \in A$, set

$$\mathbb{S}_{\overline{\alpha}} = \left\{ p_1^{\alpha_1}, p_1^{\alpha_1} p_2^{\alpha_2}, \cdots, \prod_{j=1}^{n} p_j^{\alpha_j}, \cdots \right\} \subseteq \mathbb{N}.$$

Note that elements in the list need not be distinct. However, each $\mathbb{S}_{\bar{\alpha}}$ is infinite because $\bar{\alpha}$ contains infinitely many 1-s. Given any $\bar{\alpha}, \bar{\beta} \in A$ with $\bar{\alpha} \neq \bar{\beta}$, let j denote the smallest integer such that $\alpha_j \neq \beta_j$. Suppose without loss of generality that $\alpha_j = 1$ and $\beta_j = 0$. This means every entry in $\mathbb{S}_{\bar{\alpha}}$ starting from the j-th member of its list is a multiple of p_j. None of the corresponding elements of $\mathbb{S}_{\bar{\beta}}$ has this property. Thus $\mathbb{S}_{\bar{\alpha}} \cap \mathbb{S}_{\bar{\beta}}$ is the finite set

$$\left\{ \prod_{k=1}^{\ell} p_k^{\alpha_k} : 1 \leq \ell \leq j-1 \right\}.$$

□