Math 320 Assignment 9
Due Wednesday, November 21 at start of class

Instructions

(i) Solutions should be well-crafted, legible and written in complete English sentences. You will be graded both on accuracy as well as the quality of exposition.

(ii) Theorems stated in the text and proved in class do not need to be reproved. Any other statement should be justified rigorously.

(iii) If your assignment has more than one page, staple them together.

(iv) Do not forget to include your name and SID.

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1. Let $p \geq 0$. Determine whether $\sum_n a_n$ converges or diverges for each of the following choices of a_n (the answer can depend on the value of p):

$$\text{(a) } a_n = \frac{\sqrt{n+1} - \sqrt{n}}{n^p}, \qquad \text{(b) } a_n = \frac{1}{(\log n)^{n^p}}, \qquad \text{(c) } a_n = \Big(\frac{n}{n+1}\Big)^{n^2}.$$

2. Follow the steps outlined below to give a proof of the alternating series test. Suppose that $a_1 \geq a_2 \geq a_3 \geq \cdots \geq 0$ and that $\lim_{n\to\infty} a_n = 0$. Let $s_n = \sum_{k=1}^{n} (-1)^k a_k$.

   (a) Prove that if $n > m \geq 0$ then $|s_n - s_m| \leq a_{m+1}$.

   (b) Prove that $\sum_{k=1}^{\infty} (-1)^k a_k$ converges and that, for all $n \geq 0$,

$$\Big|\sum_{k=1}^{\infty} (-1)^k a_k - s_n\Big| \leq a_{n+1}.$$

   This shows that for this special sort of alternating series, the error in approximating the infinite sum by a partial sum is at most the first omitted term.

3. (a) Determine all complex values of z for which $\sum_{n=0}^{\infty} \frac{(-1)^n}{(n+1)^2} z^n$ converges.

   (b) Find a power series $\sum_{n=0}^{\infty} a_n z^n$ that converges for all complex z with $|z| < 1$ and diverges for all $|z| \geq 1$.

   (c) Find a power series $\sum_{n=0}^{\infty} b_n z^n$ that converges for at least one complex number z with $|z| = 1$ and diverges for at least one z with $|z| = 1$.

4. Determine the radius of convergence of the power series $\sum_{n=1}^{\infty} a_n z^n$ for each of the following.

   (a) $a_n = (1 + 1/n)^{-n^2}$,

   (b) $a_{2m} = m^2$, $a_{2m+1} = 4^{-m}$,

   (c) $a_{2m} = (\log m)^m$, a_{2m+1} arbitrary.

5. Suppose that $\sum_n a_n z^n$ and $\sum_n b_n z^n$ have radii of convergence R_1 and R_2, respectively. Prove that the radius of convergence of $\sum_n a_n b_n z^n$ is at least $R_1 R_2$.

6. If $\sum_n a_n z^n$ has radius of convergence R, what are the radii of convergence of $\sum_n a_n z^{2n}$ and $\sum_n a_n^2 z^n$?

7. For each of the following series, determine which values of z give convergence and which give divergence.

$$\text{(a) } \sum_{n=0}^{\infty} \Big(\frac{z}{1+z}\Big)^n, \qquad \text{(b) } \sum_{n=0}^{\infty} \frac{z^n}{1+z^{2n}}.$$