**Math 440/508 Quiz 3 Solution**

**Name:** **SID #:**

1. Evaluate
$$\int_\gamma \frac{dz}{z},$$
where $\gamma : [0,1] \to \mathbb{C}$ is the closed polygon joining the points
$$1-i,\ 2,\ 1+i,\ 2i, -1+i, -2,\ -1-i,\ -2i,\ 1-i$$
in that order. Give complete justification. (*Hint: Avoid direct computation if possible.*)
(10 points)

*Proof.* The polygon $\gamma$ is a square (consisting of four line segments, one in each quadrant) oriented in the counterclockwise direction and passing through the points $\pm 2$ and $\pm 2i$. Denote by $\gamma_i$ the straight line segment of the rhombus lying in the $i$-th quadrant. Define $\Gamma = \Gamma_1 \cup \Gamma_2 \cup \Gamma_3 \cup \Gamma_4$ to be the circle of radius 2, with $\Gamma_i$ denoting the portion of $\Gamma$ that lies in the $i$-th quadrant. Note that for each $i$, the curves $\gamma_i$ and $\Gamma_i$ have the same endpoints.

Our claim is that
$$\oint_{\gamma_i} \frac{dz}{z} = \oint_{\gamma_i} \frac{dz}{z}, \qquad i = 1, \cdots, 4. \tag{1}$$
Assuming the claim for a moment, we see that
$$\begin{aligned}\oint_\gamma \frac{dz}{z} &= \sum_{i=1}^4 \oint_{\gamma_i} \frac{dz}{z} \\ &= \sum_{i=1}^4 \oint_{\Gamma_i} \frac{dz}{z} \\ &= \oint_\Gamma \frac{dz}{z} = \int_0^{2\pi} \frac{2ie^{i\theta}}{2e^{i\theta}}\, d\theta = 2\pi i.\end{aligned}$$

It remains to prove (1). Fix $i$. Let $\mathbb{D}$ be any disk not containing the origin but containing $\gamma_i$ and $\Gamma_i$. Then $\gamma_i - \Gamma_i$ is a closed curve in $\mathbb{D}$. (To clarify, the notation $\gamma_i - \Gamma_i$ represents the curve that goes from $\gamma_i(0)$ to $\gamma_i(1)$ via $\gamma_i$ in the time interval $[0, 1/2]$ and returns to $\gamma_i(1)$ via $\Gamma_i$ in $[1/2, 1]$). Since the function $z \mapsto 1/z$ is holomorphic on $\mathbb{D}$, Cauchy's theorem for the disk yields
$$\int_{\gamma_i - \Gamma_i} \frac{dz}{z} = 0, \qquad \text{i.e.,} \int_{\gamma_i} \frac{dz}{z} = \int_{\Gamma_i} \frac{dz}{z},$$
which is the desired conclusion. $\square$