Chapter 3, Exercise 14

Certain sets have geometric properties that guarantee they are simply connected.

- A subset $\Omega \subset \mathbb{C}$ is **convex** if for any two points in Ω, the straight line segment between them is contained in Ω. Prove that a convex open set is simply connected.
- More generally, an open set $\Omega \subset \mathbb{C}$ is **star-shaped** if there exists a point $z_0 \in \Omega$ such that for any $z \in \Omega$, the straight line segment between z and z_0 is contained in Ω. Prove that a star-shaped open set is simply connected. Conclude that the slit plane $\mathbb{C} - \{(-\infty, 0]\}$ (and more generally any sector, convex or not) is simply connected.
- What are other examples of open sets that are simply connected?

Solution

Part a

First of all, it is clear that Ω is path-connected: a path connecting x and y in Ω is given by the straight line segment connecting x and y, which is contained in Ω by convexity. Let $x, y \in \Omega$, and let $\gamma_1(t)$ and $\gamma_2(t)$ be two curves in Ω with $\gamma_1(0) = \gamma_2(0) = x$ and $\gamma_1(1) = \gamma_2(1) = y$. Then we can define the homotopy

$$\Gamma(s, t) = s\gamma_2(t) + (1 - s)\gamma_1(t).$$

This is a continuous function of s and t, and each point $\Gamma(s, t)$ lies on the line segment connecting $\gamma_1(t)$ and $\gamma_2(t)$, and is therefore in Ω by convexity. Clearly $\Gamma(s, 0) = x$ and $\Gamma(s, 1) = y$ for all s. Thus Γ is a fixed-endpoint homotopy between the curves γ_1 and γ_2, and Ω is simply connected.

Part b

As in part a, it is clear that Ω is path-connected: if $x, y \in \Omega$, then the concatenation of the straight line segment connecting x to z_0 and the straight line segment connecting z_0 to y is contained in Ω.

Let $x \in \Omega$ and let γ be a closed curve with $\gamma(0) = \gamma(1) = x$. We need to show that γ can be contracted to a point in Ω. Defining

$$\Gamma(s, t) = sz_0 + (1 - s)\gamma(t),$$

we have that $\Gamma(s, 0) = \Gamma(s, 1)$ for all s, and so the curve $\Gamma(s, t)$ remains closed for all s. Γ is clearly continuous, and $\Gamma(s, t) \in \Omega$ for all s and t by the star-shape of Ω. Therefore, Ω is simply connected.

It's easy to see that the split complex plane is star-shaped: any point on the real line is a witness. Similarly, any sector of the complex plane is star-shaped, any point lying along the bisector of the sector in Ω is a witness.

Part c

Answers may vary. I did not award points to students who provided a star-shaped region as an example.

Chapter 8, Exercise 5

Prove that $f(z) = -\frac{1}{2}(z + 1/z)$ is a conformal map from the half-disc $\{z = x + iy : |z| < 1, y > 0\}$ to the upper half-plane. [Hint: The equation $f(z) = w$ has two distinct roots in $\mathbb{C}$ whenever $w \neq \pm 1$. This is certainly the case if $w \in \mathbb{H}$.]

Solution

First, we need to verify that f maps the half-disc into the upper half-plane. Let z be a point in the upper half-disc. Then z has imaginary part bounded above by 1 and $\frac{1}{z}$ has imaginary part bounded above by -1. Thus $z + 1/z$ has negative imaginary part, and $-\frac{1}{2}\left(z + \frac{1}{z}\right)$ has positive imaginary part.

Next, we verify that f is bijective. Suppose that $f(z) = w$. Then $z^2 + 2wz + 1 = 0$ by an easy algebraic calculation. For any fixed w in the upper half-plane this equation has exactly two distinct complex roots in $\mathbb{C}$.

Let z_1 and z_2 be these roots. By Viéta's formulas, we have that $z_1 z_2 = 1$, and $z_1 + z_2 = -2w$. Since $z_1 z_2 = 1$, we have that either z_1 and z_2 both lie on the unit circle, or exactly one of z_1 and z_2 lies in the open unit disc. But if z_1 lies on the unit disc, then $z_2 = \overline{z_1}$, so $z_1 + z_2$ is real and cannot be equal to $-2w$ for any w in the upper half-plane. Thus exactly one of z_1 and z_2 lies in the open unit disc. Without loss of generality assume it is z_1. Notice that we have that z_1 is the only root of $z_1 + z_2 = -2w$ that lies in the unit disc, so it immediately follows that $f(z)$ is an injective function. If we can show that z_1 lies in the upper half of the disc, this will be sufficient to show that f is surjective.

For this we use $z_1 + z_2 = -2w$. If z_1 lies in the lower half of the disc, then z_1 has imaginary part bounded below by -1, and z_2 has imaginary part bounded below by 1. Thus $z_1 + z_2$ has positive imaginary part, but $-2w$ has negative imaginary part. This contradiction establishes that z_1 must not lie in the lower half of the disc. If z_1 lies on the real axis, then z_1 and z_2 are both real, so their sum cannot be $-2w$. Thus z_1 lies in the upper half-disc, and $f(z)$ is bijective.

Finally, we verify that f is holomorphic in the upper half-disc. But this is clear because the only pole of f occurs at $z = 0$. Thus f is conformal.

Chapter 8, Exercise 15

Here are two properties enjoyed by automorphisms of the upper half-plane.

- Suppose Φ is an automorphism of $\mathbb{H}$ that fixes three distinct points on the real axis. Then Φ is the identity.

- Suppose (x_1, x_2, x_3) and (y_1, y_2, y_3) are two pairs of distinct points on the real axis with $x_1 < x_2 < x_3$ and $y_1 < y_2 < y_3$. Prove that there exists a unique automorphism Φ of $\mathbb{H}$ such that $\Phi(x_j) = y_j$, $j = 1, 2, 3$. The same conclusion holds if $y_3 < y_1 < y_2$ or if $y_2 < y_3 < y_1$.

Solution

Part a

We know that each automorphism of the upper half-plane is a Möbius transformation of the form

$$f(z) = \frac{az + b}{cz + d}$$

where the matrix

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

can be chosen to have determinant equal to 1. Let x_1, x_2, x_3 be distinct points on the real axis that are fixed by Φ. Then the equation $\Phi(x) = x$ has 3 real solutions. But this equation can be rewritten

$$cx^2 + (d - a)x - b = 0$$

which has at most 2 real solutions unless $c = b = 0$ and $d = a$. Thus Φ is the identity automorphism.

Part b

The cross-ratio

$$\frac{z - z_1}{z - z_3} \frac{z_2 - z_3}{z_2 - z_1}$$

is a Möbius transformation that maps z_1 to 0, z_2 to 1, and z_3 to ∞. Let Φ_1 be the cross-ratio for x_1, x_2, x_3 and Φ_2 be the cross-ratio for y_1, y_2, y_3. Then

$$\Phi_2^{-1} \circ \Phi_1(z)$$

is a Möbius transformation that maps x_1 to y_1, x_2 to y_2, and x_3 to y_3. We need to verify that this Möbius transformation maps the upper half-plane to itself.

Evidently this transformation maps the real axis to itself: it is a Möbius transformation so the real axis must map to a line or circle, and 3 points are sufficient to determine a line or circle. Thus the upper half-plane is mapped to either the upper or lower half-plane according to the sign of the determinant of the Möbius transformation. This can be determined through either a direct calculation or common sense: since $x_1 < x_2 < x_3$, $y_1 < y_2 < y_3$ and Möbius transformations preserve "handedness", it follows that the upper half-plane maps to the upper half-plane. The same argument works for the case where $y_2 < y_3 < y_1$ or $y_3 < y_1 < y_2$.

The uniqueness of this automorphism follows from part a and a standard group-theoretic argument: if two such automorphisms Φ_1 and Φ_2 exist, then $\Phi_2^{-1} \circ \Phi_1$ is the identity, proving that Φ_1 and Φ_2 are the same.