Math 440/508 Take-Home Final, Fall 2017

- *The final exam is due by noon on Tuesday December 5. It can be either emailed as a pdf to the instructor, or left under the instructor's office door in Math 214. Late submissions will not be accepted.*
- *Your work will be graded both on mathematical content and quality of exposition. Please pay attention to the presentation standards.*
- *Feel free to ask the instructor for hints and clarification, but please do not discuss the exam with each other.*
- *Solutions should be based on the material covered in the course. If you need additional results, state and prove them clearly or provide adequate references. Such solutions will be accepted only if the methodology does not lie significantly beyond the scope of the course.*

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1. Using contour integration, show that
$$\int_0^\infty \frac{x^a}{1+x^b}\,dx = \frac{\pi}{b\sin\left(\frac{a+1}{b}\pi\right)}.$$
Here a and b are integers, with $0 \le a \le b-2$.

(20 points)

2. Let $\mathbb{D}$ denote the open unit disk. Suppose that $F : \mathbb{D} \to \mathbb{D}$ is a function such that for any three points in $\mathbb{D}$, there exists an analytic function on $\mathbb{D}$, possibly depending on these points and bounded above by 1 in absolute value, which agrees with F at these three points. Is it true that F itself is analytic?

(20 points)

3. Give brief answers to the following questions, with adequate justification.

($15 \times 4 = 60$ points)

(a) Let $A(r, R) := \{z : r \le |z| \le R\}$ and $\mathcal{C}_r = \{z : |z| = r\}$. Does there exist a surjective holomorphic map from $A(1, R^2)$ to $A(1, R)$, that carries $\mathcal{C}_1$ onto itself and $\mathcal{C}_{R^2}$ onto $\mathcal{C}_R$?

(b) Fix any $\rho > 0$. Show whether the function
$$f_n(z) = 1 + \frac{1}{z} + \frac{1}{2!z^2} + \cdots + \frac{1}{n!z^n}$$
has all its zeros inside the circle $|z| = \rho$ if n is sufficiently large.

(c) Can you find a multiply connected domain G so that every nonvanishing analytic function on G admits an analytic square root?

(d) Let $A = \{z : r < |z| < 1\}$ (where $r > 0$) and B denote the domain bounded by the two circles $|z - \frac{1}{4}| = \frac{1}{4}$ and $|z| = 1$. A fractional linear transformation maps A onto B. Find r.