Homework 3 - Math 440/508, Fall 2014

Due Monday November 17 at the beginning of lecture.

Instructions: Your homework will be graded both on mathematical correctness and quality of exposition. Please pay attention to the presentation of your solutions.

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1. We have seen in class that the family of circles in $\mathbb{C}_\infty$ remains invariant under Möbius transformations. This problem provides more refined information about certain properties related to a circle that are preserved by a Möbius transformation.
   (a) Let Γ be a circle in $\mathbb{C}_\infty$ through points z_2, z_3, z_4. The points z, z^* in $\mathbb{C}_\infty$ are said to be *symmetric* about Γ if
   $$(z^*, z_2, z_3, z_4) = \overline{(z, z_2, z_3, z_4)},$$
   where $(\cdot, \cdot, \cdot, \cdot)$ denotes the cross-ratio. Verify the *symmetry principle*: If a Möbius transformation T takes a circle Γ_1 onto the circle Γ_2, then any pair of points symmetric with respect to Γ_1 are mapped by T onto a pair of points symmetric about Γ_2.

   (b) If Γ is a circle then an *orientation* for Γ is an ordered triple of points (z_2, z_3, z_4) such that each $z_j \in \Gamma$, $j = 2, 3, 4$. If (z_2, z_3, z_4) is an orientation of Γ then we define the right hand side of Γ with respect to this orientation to be $\{z : \text{Im}(z, z_2, z_3, z_4) > 0\}$. Prove the *orientation principle*: Let Γ_1 and Γ_2 be two circles in $\mathbb{C}_\infty$ and let T be a Möbius transformation such that $T(\Gamma_1) = \Gamma_2$. Let (z_2, z_3, z_4) be an orientation for Γ_1. Then T takes the right (respectively left) hand side of Γ_1 to the right (respectively left) hand side of Γ_2 with respect to the orientation (Tz_2, Tz_3, Tz_4).

2. *This problem concerns a principle of analytic continuation known as the Schwarz reflection principle. Please read §5.4 of Chapter 2 of the textbook before attempting this question.*

   (a) Use the symmetry principle used in Theorems 5.5 and 5.6 in Chapter 2 of the textbook to obtain a version of the Schwarz reflection principle for the unit disc. More precisely, suppose that f is a holomorphic function on $\mathbb{D}$ which is nonvanishing on $\mathbb{D} \setminus \{0\}$, continuous up to the boundary, with $f(\partial\mathbb{D}) \subseteq \partial\mathbb{D}$. Show that f can be continued analytically as an entire function.

   (b) Show that an analytic function in $\overline{\mathbb{D}}$ which assumes a constant modulus on the boundary must be a rational function.

   (c) Does the statement in part (b) remain valid if f is assumed to be analytic in $\mathbb{D}$? Give reasons for your answer.

3. The aim of this problem is to describe the automorphism group of an annulus, and in the process also answer the following question: when are two annuli conformally equivalent? The *modulus* of an annulus $\{z : a < |z - z_0| < b\}$ with inner radius a and outer radius b is defined to be $\frac{1}{2\pi}\log\left(\frac{b}{a}\right)$.
   (a) Show that any conformal map from one annulus centred at the origin to another such annulus extends to a conformal self-map of the punctured plane.
   (b) Show that there is a conformal map of one annulus onto another if and only if the annuli have the same moduli.
   (c) Show that any automorphism of the annulus $\{z : a < |z| < b\}$ is either a rotation $z \mapsto e^{i\varphi}z$ or a rotation followed by the inversion $z \mapsto ab/z$.

4. Show that any open connected set $\Omega \subseteq \mathbb{C}_\infty$ whose boundary in $\mathbb{C}_\infty$ consists of two disjoint circles in $\mathbb{C}_\infty$ can be mapped by a Möbius transformation to $\Omega' = \{z : r < |z| < 1\}$ for a unique $r \in (0,1)$.

5. Prove Vitali's theorem: Suppose that G is an open connected set. Assume that there is a locally bounded collection $\{f_n\} \subseteq \mathbb{H}(G)$ and a function $f \in \mathbb{H}(G)$ such that the set
$$A = \{z \in G : \lim_{n\to\infty} f_n(z) = f(z)\}$$
has a limit point in G. Show that $f_n \to f$ in $\mathbb{H}(G)$.

6. Let $\mathbb{D}$ denote the open unit disc. Show that $\mathcal{F} \subseteq \mathbb{H}(\mathbb{D})$ is normal if and only if there is a sequence $\{M_n\}$ of positive constants with the following properties:
   (a) $\limsup_{n\to\infty} M_n^{\frac{1}{n}} \leq 1$,
   (b) If $f(z) = \sum_{n=0}^{\infty} a_n z^n \in \mathcal{F}$, then $|a_n| \leq M_n$ for all n.