Homework 1 - Math 440/508, Fall 2014

Due Friday September 19 at the beginning of lecture.

Instructions: Your homework will be graded both on mathematical correctness and quality of exposition. Please pay attention to the presentation of your solutions.

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1. In each of the two problems below, $\mathcal{F}_i$ is a class of entire functions. Characterize all functions that lie in $\mathcal{F}_i$.
   (a) The collection $\mathcal{F}_1$ consists of all functions f such that the power series expansion of f about every point in $\mathbb{C}$ has at least one vanishing coefficient.
   (b) The collection $\mathcal{F}_2$ consists of functions f obeying the inequality
   $$|f(z)| \leq A + B|z|^k \qquad \text{for all } z \in \mathbb{C},$$
   where A, B and k are fixed positive numbers, possibly depending on f but independent of z.

2. If f is analytic on an open set D, show that $\nabla\big(\mathrm{Re}(f)\big)$ is obtained by rotating $\nabla\big(\mathrm{Im}(f)\big)$ by 90°. Here ∇ denotes the two-dimensional gradient vector.

3. Let f be an analytic function on a open connected set D. In each of the two problems below, show that f is constant.
   (a) The function $\overline{f}$ is also analytic on D.
   (b) $|f|$ is constant on D.

4. Determine which of the following functions are analytic and at which points of D:
   (a) $f(z) = \mathrm{Re}(z)$, $D = \mathbb{C}$;
   (b) $f(z) = \mathrm{Im}(z)$, $D = \mathbb{C}$;
   (c) $f(z) = az^2 + bz\overline{z} + c\overline{z}^2$, where a, b, c are constants, $D = \mathbb{C}$;
   (d) $f(z) = \sin x \sinh y + i \cos x \cosh y$, $D = \mathbb{C}$;
   (e)
   $$H(z) = \int_0^1 \frac{h(t)}{t - z}, \qquad D = \mathbb{C} \setminus [0, 1],$$
   where h is a continuous, complex-valued function on $[0, 1]$.

5. (a) If $\gamma : [0, 1] \to \mathbb{C}$ is a continuously differentiable closed curve and $a \notin \gamma[0, 1]$, then show that the function
   $$n_\gamma(a) = \frac{1}{2\pi i} \oint_\gamma \frac{dz}{z - a}$$

is holomorphic on $\mathbb{C} \setminus \{\gamma\}$, which is in fact integer-valued everywhere in its domain of definition.

(b) The quantity $n_\gamma(a)$ is called the *index* or *winding number* of γ with respect to the point a. Compute the winding number of $\gamma(t) = a + e^{2\pi i n t}$, $t \in [0, 1]$ with respect to a to convince yourself that the nomenclature is justified.

(c) Give an example of a closed curve γ in $\mathbb{C}$ of finite length such that for any integer k there is a point $a \notin \{\gamma\}$ with $n_\gamma(a) = k$.

6. For any $\epsilon > 0$, describe the set

$$A_\epsilon = \left\{\omega : \omega = \exp\left(\frac{1}{z}\right), \quad 0 < |z| < \epsilon\right\},$$

and use it to deduce that A_ϵ is independent of ϵ.