The Arzela-Ascoli Theorem

Let (Ω, d) be a complete metric space, and G denote an open subset of $\mathbb{C}$. The notation $\mathcal{C}(G,\Omega)$ represents the class of continuous functions from G to Ω. We endow $\mathcal{C}(G,\Omega)$ with the metric space topology of uniform convergence on compact sets. Recall that if $\{K_R\}$ is an increasing family of compact subsets of G that fill out G, then a metic that generates this topology is given by

$$\rho(f,g) = \sum_{R=1}^{\infty} 2^{-R} \frac{\rho_R(f,g)}{1+\rho_R(f,g)} \quad \text{where} \quad \rho_R(f,g) = \sup\{|f(z)-g(z)| : z \in K_R\}.$$

We fix a sequence $\{K_R\}$ and its associated metric ρ, and use these without further reference in the sequel.

Theorem 0.1 (Arzela-Ascoli)**.** *A set $\mathcal{F} \subseteq \mathcal{C}(G,\Omega)$ is normal (i.e., $\overline{\mathcal{F}}$ is compact) if and only if the following two conditions are satisfied:*
(a) for each $z \in G$, $\{f(z) : f \in \mathcal{F}\}$ has compact closure in Ω;
(b) $\mathcal{F}$ is equicontinuous at each point in G.

Proof. Suppose that $\mathcal{F}$ is normal. For every fixed $z \in G$, the evaluation map $\varphi_z : f \mapsto f(z)$ is continuous from $\mathcal{C}(G,\Omega)$ to Ω (why?). Since the image of a compact set under a continuous map is compact, $\varphi_z(\overline{\mathcal{F}})$ is compact. In particular, $\{f(z) : f \in \mathcal{F}\}$ is contained in $\varphi_z(\overline{\mathcal{F}})$, and hence has compact closure.

We now proceed to prove (b); i.e., for any $z_0 \in G$ and any $\epsilon > 0$, we aim to find $\delta > 0$ such that for all $f \in \mathcal{F}$,

$$(1) \qquad |f(z) - f(z_0)| < \epsilon \quad \text{whenever} \quad |z - z_0| < \delta.$$

First fix a small closed disc $\overline{B(z_0;r)} \subseteq G$. Since this closed disc is compact, it is contained in one of the compact sets K_R for some $R \geq 1$. Let us consider the collection of open balls $\{\mathbb{B}_\rho(g; 2^{-R}\frac{\frac{\epsilon}{3}}{1+\frac{\epsilon}{3}}) : g \in \mathcal{F}\}$, which forms an open cover of $\overline{\mathcal{F}}$. Here $\mathbb{B}_\rho(g;\epsilon) = \{h \in \mathcal{C}(G,\Omega) : \rho(g,h) < \epsilon\}$. Since $\mathcal{F}$ is relatively compact, we may extract a finite subcover. In other words, there exists finitely many functions $f_1, f_2, \cdots, f_N \in \mathcal{F}$ such that

$$(2) \qquad \mathcal{F} \subseteq \bigcup_{k=1}^{N} B_\rho(f_k; 2^{-R}\frac{\epsilon}{3+\epsilon}).$$

Now choose $0 < \delta \leq r$ depending on ϵ so that for all $1 \leq k \leq N$,

$$(3) \qquad |f_k(z) - f_k(z_0)| < \frac{\epsilon}{3} \quad \text{whenever} \quad |z - z_0| < \delta.$$

We will show that (1) holds for this choice of δ and all $f \in \mathcal{F}$.

The inclusion (2) implies that for any $f \in \mathcal{F}$, there exists f_k for some $1 \leq k \leq N$ such that $\rho(f, f_k) < 2^{-R}\frac{\epsilon}{3+\epsilon} = 2^{-R}\frac{\epsilon/3}{1+\epsilon/3}$. Since $\rho(f, f_k) \geq 2^{-R}\frac{\rho_R(f,f_k)}{1+\rho_R(f,f_k)}$, we deduce that $\rho_R(f, f_k) < \frac{\epsilon}{3}$. Combining this with the choice of δ made in (3), we obtain that for any $f \in \mathcal{F}$,

$$\begin{aligned}|f(z) - f(z_0)| &\leq |f(z) - f_k(z)| + |f(z_0) - f_k(z_0)| + |f_k(z) - f_k(z_0)| \\ &\leq 2\rho_R(f, f_k) + \frac{\epsilon}{3} < \frac{2\epsilon}{3} + \frac{\epsilon}{3} = \epsilon.\end{aligned}$$

Thus $\mathcal{F}$ is equicontinuous, as claimed. This concludes the proof of the "only if" part.

Conversely, suppose that conditions (a) and (b) hold. To prove that $\mathcal{F}$ is normal, it suffices to show that every sequence in $\mathcal{F}$ has a subsequence that converges in $\mathcal{C}(G, \Omega)$. This in turn follows if the subsequence is Cauchy, since $\mathcal{C}(G, \Omega)$ is complete. Given $\{f_n : n \geq 1\} \subseteq \mathcal{F}$, we first extract a subsequence as follows. Let $\{z_j : j \geq 1\}$ be an enumeration of $G \cap (\mathbb{Q} + i\mathbb{Q})$, i.e., the points in G with rational real and imaginary parts. By hypotheses (a) and the Cantor diagonalization process, we can find a subsequence $\{f_{n_k}\} \subseteq \{f_n\}$ such that $\lim_{k\to\infty} f_{n_k}(z_j)$ exists for all $j \geq 1$. Set $\omega_j = \lim_{k\to\infty} f_{n_k}(z_j) \in \Omega$. We now proceed to show that $\{g_k = f_{n_k}\}$ is a Cauchy sequence.

Fix $\epsilon > 0$. Choose $R \geq 1$ large enough so that $2^{-R} < \frac{\epsilon}{2}$. Since every function in $\mathcal{F}$ is uniformly continuous on each K_r, it follows from hypothesis (b) that there exists $\delta > 0$ such that

$$|f(z) - f(z')| < \frac{\epsilon}{8} \quad \text{for all } f \in \mathcal{F} \text{ and } z, z' \in K_{R+1} \text{ with } |z - z'| < \delta. \tag{4}$$

Cover the compact subset K_R by finitely many balls of radius $\delta/2$, and pick a point $z_j \in (\mathbb{Q} + i\mathbb{Q}) \cap G$ from each ball. Let $L \geq 1$ be chosen large enough so that for every chosen z_j,

$$|g_k(z_j) - g_\ell(z_j)| \leq |g_k(z_j) - \omega_j| + |g_\ell(z_j) - \omega_j| < \frac{\epsilon}{8} + \frac{\epsilon}{8} = \frac{\epsilon}{4} \quad \text{for all } k, \ell \geq L. \tag{5}$$

For such k and ℓ, it follows from (4) and (5) that for all $z \in K_R$,

$$\begin{aligned}|g_k(z) - g_\ell(z)| &\leq |g_k(z) - g_k(z_j)| + |g_\ell(z) - g_\ell(z_j)| + |g_k(z_j) - g_\ell(z_j)| \\ &\leq \frac{\epsilon}{8} + \frac{\epsilon}{8} + \frac{\epsilon}{4} = \frac{\epsilon}{2},\end{aligned}$$

where z_j is chosen to lie in the same $\delta/2$-ball as z. The last inequality implies that $\rho_r(g_k, g_\ell) \leq \rho_R(g_k, g_\ell) < \frac{\epsilon}{2}$ for $1 \leq r \leq R$, hence

$$\rho(g_k, g_\ell) = \left[\sum_{r=1}^{R} + \sum_{r=R+1}^{\infty}\right] 2^{-r}\frac{\rho_r(g_k, g_\ell)}{1 + \rho_r(g_k g_\ell)} < \rho_R(g_k, g_\ell) + \frac{\epsilon}{2} < \epsilon$$

for all $k, l \geq L$. This proves the desired claim. $\square$