Homework 3 - Math 541, Fall 2012

Due Wednesday November 7 at the beginning of lecture.

Instructions: Your homework will be graded both on mathematical correctness and quality of exposition. Please pay attention to the presentation of your solutions.

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1. Determine whether the following statements are true or false. Give brief justification for your answer.
   (a) There exists $f \in L^1(\mathbb{T})$ such that $S_N f$ does not tend to f in L^1 norm as $N \to \infty$.
   (b) The Fourier transform is bounded from $L^p(\mathbb{R}^d)$ to $L^{p'}(\mathbb{R}^d)$ for every $1 \le p \le 2$.
   (c) The Fourier transform is bounded from $L^p(\mathbb{R}^d)$ to $L^{p'}(\mathbb{R}^d)$ for some $p > 2$.

2. Recall that the *total variation* $V(f, I)$ of a function f over an interval $I = (a, b) \subseteq \mathbb{R}$ is

$$V(f, I) = \sup_P \sum_{j=1}^{k} |f(t_j) - f(t_{j-1})|,$$

   where the supremum is taken over all partitions $P = \{(t_0, t_1, \cdots, t_k) : a < t_0 < t_1 < \cdots < t_k < b\}$. We denote $V(f) = V(f, \mathbb{R})$ if f is globally defined, and say that f is of bounded variation, denoted $f \in BV(\mathbb{R})$, if $V(f) < \infty$.
   If $f \in BV(\mathbb{R})$ and has compact support, or if $f \in BV(\mathbb{T})$, show that

$$\left|\widehat{f}(\xi)\right| \le 2\pi V(f)|\xi|^{-1} \quad \text{for all } \xi \ne 0.$$

   Convince yourself that the proofs should be essentially identical for the two cases, and present just one.

3. The aim of this problem is to show that if $f \in C(\mathbb{T})$ has bounded variation, then $S_N f \to f$ uniformly as $N \to \infty$. Fill in these steps to arrive at this conclusion.
   (a) Let $g \in L^1(\mathbb{T})$. Suppose that the Fourier coefficients of g satisfy the property that for every $\epsilon > 0$ there is $\lambda = \lambda(\epsilon) > 1$ such that

$$\limsup_{n\to\infty} \sum_{n \le |k| \le \lambda n} |\widehat{g}(k)| < \epsilon.$$

   Show that $S_n(g)(t)$ converges if and only if $\sigma_n(g)(t)$ does.
   (b) Use Problem 2 and the result in part (a) to prove the desired convergence of $S_N f$ if $f \in C(\mathbb{T}) \cap BV(\mathbb{T})$.

4. We proved in class that there exists $f \in C(\mathbb{T})$ whose Fourier series fails to converge uniformly, in fact, diverges to infinity at a point. Use the strategy outlined below (due to Salem) to prove a result of Pál and Bohr, which provides an interesting counterpoint: for any real-valued $f \in C(\mathbb{T})$, there exists a homeomorphism, in fact a continuous strictly increasing function $\varphi : \mathbb{T} \to \mathbb{T}$ with $\varphi(0) = 0, \varphi(2\pi) = 2\pi$, such that $S_N(f \circ \varphi) \to f \circ \varphi$ uniformly on $\mathbb{T}$.
   (a) Argue that without loss of generality f can be chosen to have mean value zero and to vanish at the endpoints 0 and 2π. Deduce from this that there exists $0 < a < 2\pi$ such that $f(a) = 0$.
   (b) Suppose first that the point a defined in part (a) is unique. Prove the Pál-Bohr theorem under this restricted assumption. (Hint: You may use without proof the following version of the Riemann mapping theorem: Let Ω be a domain in the plane bounded by a simple closed curve γ. Then there exists a conformal mapping (i.e. a holomorphic bijection) of the unit disc $\mathbb{D} = \{|z| < 1\}$ onto Ω. The power series representing any such mapping converges uniformly on $\partial\mathbb{D}$ and determines a continuous bijection of $\partial\mathbb{D}$ onto γ.)
   (c) Suppose now that a is not the only point on $(0, 2\pi)$ where f vanishes. Let $t_1 \in I_1 = (0, a)$ and $t_2 \in I_2 = (a, 2\pi)$ be two points such that $|f(t_i)| = \max\{|f(t)| : t \in I_i\}$. Consider the function $\omega(t)$ defined as follows,

$$\omega(t) = \left\{ \begin{array}{ll} \max\limits_{0 \le s \le t} |f(s)| + \sin\left(\dfrac{\pi t}{a}\right) & \text{if } 0 \le t \le t_1 \\ \max\limits_{t \le s \le a} |f(s)| + \sin\left(\dfrac{\pi t}{a}\right) & \text{if } t_1 \le t \le a \\ -\max\limits_{a \le s \le t} |f(s)| - \sin\left(\dfrac{\pi(t-a)}{2\pi - a}\right) & \text{if } a \le t \le t_2 \\ -\max\limits_{t \le s \le 2\pi} |f(s)| - \sin\left(\dfrac{\pi(t-a)}{2\pi - a}\right) & \text{if } t_2 \le t \le 2\pi. \end{array} \right\}$$

   Show that ω is of bounded variation and that $f + \omega$ satisfies the hypothesis of part (b).
   (d) Use part (c) and problem 3 to complete the proof of the theorem.