Homework 2 - Math 541, Fall 2012

Due Wednesday October 17 at the beginning of lecture.

Instructions: Your homework will be graded both on mathematical correctness and quality of exposition. Please pay attention to the presentation of your solutions.

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1. Determine whether each of the following statements is true or false. Give brief justification for your answers.
   (a) For any $p \in [1, \infty)$, the set of all functions in $L^p(\mathbb{R}^d)$ whose Fourier transforms have compact support is dense in $L^p(\mathbb{R}^d)$.
   (b) There exists $p \neq 2$ such that the Fourier transform can be extended as a bounded linear map from $L^p(\mathbb{R}^d)$ to $L^p(\mathbb{R}^d)$.
   (c) There exists $f \in L^\infty \setminus \cup_{1 \leq p < \infty} L^p$ whose Fourier transform is a function, not merely a distribution.
   (d) If $f \in C(\mathbb{T}^d)$ satisfies a Hölder condition with exponent α, $0 < \alpha \leq 1$, then $\widehat{f}(n) = O(|n|^{-\alpha})$.
   (e) There exists a sequence $\epsilon_n \to 0$ with the following property: for every $f \in C(\mathbb{T})$, $|\widehat{f}(n)| < \epsilon_n$ for all sufficiently large n.

2. *Evaluation of the Fourier transform for some examples.* Establish the following identities:
   (a) For any $\xi \in \mathbb{R}$,
   $$\left[(1 + x^2)^{-1}\right]\widehat{\ }(\xi) = \pi e^{-|\xi|}.$$
   (b) For any $\xi \in \mathbb{R}$,
   $$\left[e^{-x^2/2}\right]\widehat{\ }(\xi) = \sqrt{2\pi} e^{-\xi^2/2}.$$
   (c) In $\mathbb{R}^d$,
   $$\left[e^{-|\xi|}\right]\widehat{\ }(x) = (2\pi)^d \Gamma\left(\frac{d+1}{2}\right) \pi^{-\frac{d+1}{2}} \left(1 + |x|^2\right)^{-\frac{d+1}{2}}.$$
   As part of the proof of Plancherel's theorem presented in class, we showed that there exists an absolute constant $\beta > 0$ such that $(\widehat{f})^\vee = \beta^{-d} f$ almost everywhere for every $d \geq 1$ and $f \in L^1(\mathbb{R}^d)$. Use any subset of the formulae above to find the value of β.
   (*Hint for (c):* Start by writing $e^{-|\xi|}$ as
   $$e^{-|\xi|} = \pi^{-\frac{1}{2}} \int_0^\infty e^{-\frac{|\xi|^2}{4u}} e^{-u} u^{-\frac{1}{2}} \, du,$$
   then evaluate the Fourier transform using (b).)

3. Recall the definition of temperate measure introduced in class.

(a) Given any temperate measure μ and multi-index α, show that φ defined by

$$\langle \varphi, f \rangle := \int_{\mathbb{R}^d} \partial^\alpha f(x)\, d\mu(x), \quad f \in \mathcal{S}(\mathbb{R}^d) \qquad (*)$$

is a tempered distribution.

(b) Show that every element of $\mathcal{S}'(\mathbb{R}^d)$ is a finite linear combination of distributions of the form $(*)$, for certain temperate measures μ and multi-indices α.

4. This problem deals with some more examples of tempered distributions.

(a) Show that the principal value $1/x$ distribution, defined by

$$\mathrm{pv} \int f(x) x^{-1}\, dx := \lim_{\epsilon \to 0} \int_{|x|>\epsilon} f(x) x^{-1}\, dx$$

is tempered.

(b) Show that $\varphi(\cdot)$, defined by

$$\langle \varphi(z), f \rangle = \int_0^\infty x^z f(x)\, dx$$

is a holomorphic $\mathcal{S}'$-valued function of z for $\mathrm{Re}(z) > -1$, which can be continued as a meromorphic $\mathcal{S}'$-valued function on all of $\mathbb{C}$, with only simple poles at $z = -1, -2, -3, \cdots$.

5. *Poisson summation formula.* The Fourier transforms for $\mathbb{R}^d$ and $\mathbb{T}^d$ are related. An elegant application of this linkage yields the Poisson summation formula, which states that for every $f \in C_c^\infty(\mathbb{R}^d)$,

$$\sum_{n \in \mathbb{Z}^d} f(2\pi n) = (2\pi)^{-d} \sum_{n \in \mathbb{Z}^d} \widehat{f}(n).$$

Follow the steps outlined below to arrive at a proof of this formula.

(a) Given f, form the $(2\pi\mathbb{Z})^d$-periodic function $F(x) = \sum_{k \in \mathbb{Z}^d} f(x + 2\pi k)$. Argue that F equals its Fourier series.

(b) Show that

$$\widehat{F}(n) = (2\pi)^{-d} \widehat{f}(n) \quad \text{for all } n \in \mathbb{Z}^d,$$

where $\widehat{\ }$ denotes the $\mathbb{T}^d$ Fourier transform on the left and the $\mathbb{R}^d$ Fourier transform on the right.

(c) Set $x = 0$ and combine the previous steps to arrive at the Poisson summation formula.