Homework 1 - Math 541, Fall 2012

Due Friday September 28 at the beginning of lecture.

Instructions: Your homework will be graded both on mathematical correctness and quality of exposition. Please pay attention to the presentation of your solutions.

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1. (a) Show that every measurable function $f : \mathbb{T} \to \mathbb{C}$ that does not vanish identically and satisfies
$$f(x+y) = f(x)f(y) \quad \text{for all } x, y$$
is one of the maps $x \mapsto e^{inx}$, $n \in \mathbb{Z}$.
(b) Suppose that a bounded linear operator $T : L^2(\mathbb{T}^1) \to L^2(\mathbb{T}^1)$ is translation-invariant. Show that each function $\exp(inx)$ is an eigenfunction of T.

2. In applications, translation invariant operators often arise through physical symmetries. As an example, consider the problem that Fourier set out to solve: heat conduction in an infinitesimally thin, homogeneous circular wire. Rotation of the wire is the physical symmetry. Identify the wire with $\mathbb{S}^1$. Letting $u(x,t)$ be the temperature at e^{ix} at time t, this situation is modeled by
$$\partial_t u = \partial_x^2 u \text{ for } t > 0, \qquad u(x,0) = f(x), \tag{1}$$
where f is the temperature distribution at time 0.
Suppose that the problem has for each f a unique solution, as we should expect if the equation correctly models heat conduction. For each $t > 0$, the operator T_t that maps $f(x)$ to $u(x,t)$ must then be invariant under rotations of $\mathbb{S}^1$. If $f(x) = \exp(inx)$, this heuristics implies (by Problem 1(b)) that $u(x,t) = \lambda_n(t)\exp(inx)$ for some function λ_n.
(a) Find $\lambda_n(t)$.
(b) By representing a general $f \in L^2(\mathbb{S}^1)$ as the sum of its Fourier series and using the linearity of the equation, arrive at a *formal* solution to the heat equation (1). Why is this analysis merely formal?
(c) Verify, however, that the series representing u converges in $L^2(\mathbb{S}^1)$ for $t \geq 0$, and equals f when $t = 0$.
We will see later that it does indeed define a solution of (1), and does so for a larger class of initial data than merely L^2.

3. Repeat the above exercise for the wave equation in $\mathbb{T} \times [0, \infty)$. In other words, for each $f, g \in L^2(\mathbb{S}^1)$, find a formal series solution of
$$(\partial_t^2 - \partial_x^2)u(x,t) = 0, \quad u(x,0) = f(x), \quad \partial_t u(x,0) = g(x).$$

4. Show that the algebra $L^1(\mathbb{R}^d)$ has no multiplicative unit.

5. In class, we proved a result that said that any bounded linear translation-invariant operator $T : C_c(\mathbb{R}^d) \to C_b(\mathbb{R}^d)$ can be represented as $Tf = f * \mu$ for some finite Radon measure μ on $\mathbb{R}^d$). Show that the obvious analogue for $T : C_b(\mathbb{R}^d) \to C_b(\mathbb{R}^d)$ is false.

6. (a) Define the convolution of two finite measures μ and ν on $\mathbb{R}^d$, so that your definition agrees with the definition of convolution of functions $f * g$ when $\mu = f(x)\,dx$ and $\nu = g(x)\,dx$.
   (b) Show that the convolution of any two finite measures is a finite measure whose total variation satisfies

$$||\mu * \nu|| \leq ||\mu|| \cdot ||\nu||.$$

7. Recall the three properties used to define an approximate identity sequence in $\mathbb{R}^d$:

$$\int_{\mathbb{R}^d} \varphi_j \to 1 \text{ as } j \to \infty, \tag{2}$$

$$\sup_j ||\varphi_j||_1 < \infty, \tag{3}$$

$$\int_{|x|>\delta} |\varphi_j(x)|\,dx \to 0 \text{ as } j \to \infty, \text{ for all } \delta > 0. \tag{4}$$

   (a) Suppose that $\{\varphi_j\}$ is a subset of $L^1(\mathbb{R}^d)$ (not necessarily an approximate identity sequence) satisfying $||f * \varphi_j - f||_1 \to 0$ as $j \to \infty$ for every $f \in L^1$. Prove that (2) and (3) hold.
   (b) Show that there exist sequences $\{\varphi_j\} \subseteq L^1(\mathbb{R}^d)$ such that $f * \varphi_j \to f$ uniformly for all $f \in C_c(\mathbb{R}^d)$, yet (4) fails.
   (c) Likewise, show that there exist such sequences that satisfy $||f * \varphi_j - f||_1 \to 0$ for all $f \in L^1(\mathbb{R}^d)$.