Midterm - Math 440/508, Fall 2012

Due Friday November 2 at the beginning of lecture.

Instructions:

1. *You are free to use results that have been presented in class. Any other results have to be explicitly stated and proved, and as such, should lie within the scope of this course.*
2. *Your work will be graded both on mathematical correctness and quality of exposition. Please pay attention to the presentation of your solutions.*

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1. Determine whether the following results are true or false. Give brief justification for your answers.
   (a) There exists a nonconstant entire function f such that $\sqrt{f}$ is also entire.
   (b) For any polynomial p, the roots of p' lie in the convex hull of the roots of p. (Recall that the *convex hull* of a set S consists of all finite linear combinations of the form $\lambda_1 x_1 + \cdots + \lambda_n x_n$ where $x_1, x_2, \cdots, x_n$ lie in S and λ_i are non-negative numbers satisfying $\lambda_1 + \cdots + \lambda_n = 1$.
   (c) A man with a leashed dog paces around a fountain two meters in diameter. All along the dog wanders its own way, restrained only by the five foot leash. If after about twenty minutes, the man and his dog find themselves at their respective starting points, then the total number of turns around the fountain must be the same for the dog and its master.

2. Use contour integration to evaluate the following integrals:

$$\text{(a)} \int_0^\infty \frac{(\log x)^2}{x^2+x+1}\,dx \qquad\qquad \text{(b)} \oint_\gamma (2z-1)^{-3}\mathrm{Log}(1+z)\,dz,$$

   where $\gamma(t) = (2\cos t - 1)e^{it}$, $0 \le t \le 2\pi$, and "Log" denotes the principal branch of the logarithm.

3. Let f be holomorphic on an open annulus $A(a; R_1, R_2)$ centred at a with inner and outer radii R_1 and R_2 respectively, $R_1 < R_2$. The cases $R_1 = 0$ and/or $R_2 = \infty$ are allowed, and correspond respectively to the punctured disc $\mathbb{D}(a; R_2) \setminus \{a\}$ and the exterior of the disc $\mathbb{D}(a; R_1)$.
   (a) Using a keyhole contour and following the same principle as in the proof of the Cauchy integral formula, deduce rigorously the Laurent series development of f. Namely, show

that

$$f(z) = \sum_{n=-\infty}^{\infty} a_n (z-a)^n$$

where the convergence is absolute and uniform on $A(a; r_1, r_2)$ for every $R_1 < r_1 < r_2 < R_2$. Show further the coefficients a_n are given by the formula

$$a_n = \frac{1}{2\pi i} \oint_\gamma \frac{f(z)}{(z-a)^{n+1}},$$

where γ is any circle centred at a lying in $A(a; R_1, R_2)$ with anticlockwise orientation.

(b) Determine the Laurent series development of $f(z) = (z^2 - 1)^{-1}$ in the regions $A(1; 0, 2)$ and $A(0; 1, \infty)$.

4. Prove the free homotopic version of Cauchy's theorem. Namely, if f is holomorphic on an open set $\Omega \subseteq \mathbb{C}$ and if γ_0, γ_1 are two closed curves that are free homotopic in Ω, then show that

$$\int_{\gamma_0} f = \int_{\gamma_1} f.$$