Homework 2 - Math 440/508, Fall 2012

Due Wednesday October 17 at the beginning of lecture.

Instructions: Your homework will be graded both on mathematical correctness and quality of exposition. Please pay attention to the presentation of your solutions.

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1. Let L be a line in the complex plane. Suppose $f(z)$ is a continuous complex-valued function on a domain D that is analytic on $D \setminus L$. Show that $f(z)$ is analytic on D.

2. (a) Show that if $f(z)$ is an entire function and there is a nonempty disc such that $f(z)$ does not attain any values in the disc, then $f(z)$ is constant.
   (b) A function $f(z)$ on the complex plane is doubly periodic if there are two nonzero complex numbers ω_0 and ω_1 of $f(z)$ that do not lie on the same line through the origin such that $f(z+\omega_0) = f(z+\omega_1) = f(z)$ for all $z \in \mathbb{C}$. Prove that the only doubly periodic entire functions are the constants. Can you find a singly periodic non-constant entire function?

3. Evaluate the following integrals using the Cauchy integral formula:

$$\text{(a)} \oint_{|z|=1} \frac{\sin z}{z}\, dz \qquad \text{(b)} \oint_{|z|=1} \frac{dz}{z^2(z^2-4)e^z} \qquad \text{(c)} \oint_{|z-1|=2} \frac{dz}{z(z^2-4)e^z}.$$

4. Given a plane domain D, recall that a function $u : D \to \mathbb{R}$ is harmonic if $u_{xx} + u_{yy} = 0$.
   (a) If $f = u + iv$ is holomorphic on D, show that u and v are harmonic.
   (b) Two harmonic functions $u, v : D \to \mathbb{R}$ are said to be *harmonic conjugates* if $f = u + iv$ is holomorphic on D. If u is harmonic on D, show that u admits a harmonic conjugate on every disk whose closure is contained in D.
   (c) Use the Cauchy integral formula to derive the mean value property of harmonic functions, namely that

$$u(z_0) = \int_0^{2\pi} u(z_0 + \rho e^{i\theta}) \frac{d\theta}{2\pi}, \qquad z_0 \in D$$

   whenever $u(z)$ is harmonic in a domain D and the closed disc $|z - z_0| \leq \rho$ is contained in D.