Final Exam - Math 440/508, Fall 2012

Due Friday December 14 by 5pm

Instructions:

1. *Please leave your work in the instructor's mailbox or slide under her office door.*
2. *You are free to use results that have been presented in class. Any other results have to be explicitly stated and proved, and as such, should lie within the scope of this course.*
3. *Your work will be graded both on mathematical correctness and quality of exposition. Please pay attention to the presentation of your solutions.*

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1. Let $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$. Show that $\mathcal{F} \subseteq \mathcal{H}(\mathbb{D})$ is normal if and only if the following condition holds:
   There exists a sequence $\{M_n : n \geq 0\}$ of positive constants such that
   $$\limsup_{n\to\infty} M_n^{\frac{1}{n}} \leq 1$$
   and
   $$|a_n| \leq M_n \quad \text{for all } n, \quad \text{whenever } f(z) = \sum_{n=0}^{\infty} a_n z^n \in \mathcal{F}.$$

2. Suppose that Ω is a simply connected proper domain in $\mathbb{C}$ and z_0 is a fixed point in Ω. Let φ be the unique conformal mapping of Ω onto the open unit disk $\mathbb{D}$ given by the Riemann mapping theorem, normalized by the conditions $\varphi(z_0) = 0$, $\varphi'(z_0) > 0$. Show that for any analytic $f : \Omega \to \mathbb{D}$, one must have
   $$\mathrm{Re}(f'(z_0)) \leq \varphi'(z_0),$$
   with equality if and only if $f = \varphi$.

3. For each of the following domains Ω, find a conformal map w with the given specifications that maps Ω onto the upper half plane.
   (a) Fix $a > 0$. Ω is the upper half plane with a slit along the vertical interval from $z = 0$ to $z = ia$, with $w(z) \sim z$ as $|z| \to \infty$. (Hint: Start with $z \mapsto z^2$.)
   (b) Ω is the half-strip $\{-\frac{\pi}{2} < \mathrm{Re}(z) < \frac{\pi}{2},\ \mathrm{Im}(z) > 0\}$.

4. Verify whether each of the following statements is true or false, with proper justification.
   (a) For any domain $G \subset \mathbb{C}$, a collection of functions $\mathcal{F} \subset \mathcal{H}(G)$ is normal if $\mathcal{F}' = \{f' : f \in \mathcal{F}\}$ is normal.

(b) The converse statement to the one in part (a) holds.

(c) There is $f \in \mathcal{H}(\mathbb{D})$ such that $f' \neq 0$ everywhere and $f(\mathbb{D}) = \mathbb{A} = \{0 < |z| < 1\}$.

(d) (Extra credit) There is $f \in \mathcal{H}(\mathbb{A})$ such that $f' \neq 0$ everywhere and $f(\mathbb{A}) = \mathbb{D}$.