Midterm - Math 440/508, Fall 2011

Due on Monday October 24

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1. Compute the following integrals :
$$(a) \int_{-\infty}^{\infty} \frac{\sin x}{x(x-\pi)}\, dx \qquad (b) \frac{1}{2\pi i} \int_C \frac{dz}{\sin(1/z)},$$
where C is the circle $|z| = 1/5$ positively oriented.

(15 × 2 = 30 points)

2. Determine the automorphism group $\mathrm{Aut}(\mathbb{C})$ of the complex plane; i.e., the set of all one-to-one analytic maps of $\mathbb{C}$ onto $\mathbb{C}$. (Hint: examine the behavior at ∞.)

(20 points)

3. Let f be analytic on $\mathbb{C} \setminus \{0\}$, and suppose that
$$f(\{|z| = 1\}) \subseteq \mathbb{R} \quad \text{and} \quad f(z) = f(1/z) \text{ for all } z \neq 0.$$
Prove that f is real on $\mathbb{R} \setminus \{0\}$.

(20 points)

4. State whether each of the following statements is true or false. Give a short proof or a counterexample, as appropriate, in support of your claim.

(10 × 3 = 30 points)

   (a) Let Ω be an open subset of $\mathbb{R}^2$ and let $f : \Omega \to \mathbb{R}^2$ be a smooth map. Assume that f preserves orientation (i.e., the Jacobian of f is positive everywhere), and that f maps any pair of orthogonal curves to a pair of orthogonal curves. Then f must be holomorphic, after the usual identification of $\mathbb{R}^2$ with $\mathbb{C}$.

   (b) There exists a non-polynomial entire function f such that the image of every unbounded sequence under f is some unbounded sequence.

   (c) The only function that is analytic on the unit disc and satisfies
$$f''\left(\frac{1}{p}\right) + f\left(\frac{1}{p}\right) = 0 \text{ for all prime integers } p$$
is the constant zero function.