

## Practice Problem Set 2 for Midterm 2

1. For every point  $(x, y, z)$  in the first octant and lying on the ellipsoid  $x^2 + 2y^2 + 3z^2 = 6$ , consider the parallelepiped bounded by the coordinate planes and having  $(x, y, z)$  as a vertex. Find the maximum volume of the parallelepipeds thus obtained.

(Answer:  $\frac{2}{\sqrt{3}}$ )

2. Find the extreme values of the function

$$f(x, y) = 2x^2 + 3y^2 - 4x - 5$$

in the disk centered at the origin and with radius 4.

(Answer:  $\text{Max } f(-2, \pm 2\sqrt{3}) = 47, \text{ min } f(1, 0) = -7$ )

3. Given any function  $f$ ,

(a) When is the directional derivative of  $f$  a maximum?

(b) When is it a minimum?

(c) When is it 0?

(d) When is it half of its maximum value?

4. Xavier and Yolanda both have classes that end at noon and they agree to meet every day after class. They arrive at a campus cafe independently. Xavier's arrival time is  $X$  and Yolanda's arrival time is  $Y$ , where  $X$  and  $Y$  are measured in minutes after noon. The individual density functions are:

$$f_1(x) = \begin{cases} e^{-x} & \text{if } x \geq 0, \\ 0 & \text{if } x < 0 \end{cases}, \quad f_2(y) = \begin{cases} \frac{y}{50} & \text{if } 0 \leq y \leq 10, \\ 0 & \text{otherwise.} \end{cases}$$

After Yolanda arrives, she will wait at the cafe up to half an hour for Xavier and then go to the library. On the other hand, if Xavier arrives and does not find Yolanda, he will email her a message and leave immediately for the library. Find the probability that they meet at the cafe.

(Answer:  $\frac{1}{50}(1 - e^{-30})(1 - 11e^{-10}) \approx 0.020$ )

5. Sketch the solid given by the inequalities

$$\rho \leq 2, \quad \rho \leq \csc \phi.$$

6. Find the volume of the smaller wedge cut from a sphere of radius  $a$  by two planes that intersect along a diameter at an angle of  $\frac{\pi}{6}$ .

(Answer:  $\frac{\pi}{9}a^3$ )

7. Show that

$$\iiint_{\mathbb{R}^3} \sqrt{x^2 + y^2 + z^2} e^{-(x^2+y^2+z^2)} dz dy dx = 2\pi.$$

8. Evaluate the integral

$$\int_0^1 \int_{\sqrt{y}}^1 \frac{ye^{x^2}}{x^3} dx dy.$$

(Answer:  $\frac{1}{4}(e - 1)$ )

9. Evaluate the integral

$$\iint_D (x^2 + y^2)^{\frac{3}{2}} dA,$$

where  $D$  is the region in the first quadrant bounded by the lines  $y = 0$  and  $y = \sqrt{3}x$  and the circle  $x^2 + y^2 = 9$ .

(Answer:  $\frac{81\pi}{5}$ )

10. In Tuesday's class, we had a discussion about the mean and median of a random variable  $X$ . Recall that the mean is the expected value  $\mathbb{E}(X)$  of the random variable, while the median is the value  $x_0$  such that  $\mathbb{P}(X \geq x_0) = \mathbb{P}(X < x_0) = \frac{1}{2}$ . For a random variable with an exponential density function, compute the mean and the median to convince yourself that these two quantities are not equal in general. Which one is bigger for the exponential density?