Review Sheet 1

1. For what values of the number r is the function
$$f(x,y,z)=\begin{cases}\frac{(x+y+z)^r}{x^2+y^2+z^2} & \text{if } (x,y,z)\neq(0,0,0)\\ 0 & \text{if } (x,y,z)=0\end{cases}$$
continuous on $\mathbb{R}$?

(Answer: $r>2$)

2. Among all planes that are tangent to the surface $xy^2z^2=1$, find the ones that are farthest from the origin.

(Answer: $(2^{2/5})x\pm(2^{9/10})y\pm(2^{9/10})z=5$)

3. Evaluate the integral
$$\int_0^1\int_0^1 e^{\max\{x^2,y^2\}}\,dydx,$$
where $\max\{x^2,y^2\}$ means the larger of the two numbers x^2 and y^2.

(Answer: $e-1$)

4. If $f:\mathbb{R}\to\mathbb{R}$ is continuous, show that
$$\int_0^x\int_0^y\int_0^z f(t)\,dt\,dz\,dy=\frac{1}{2}\int_0^x(x-t)^2f(t)\,dt.$$

5. Recall that a function f is harmonic if $\nabla^2 f=0$.
   (a) Show that if f is a harmonic function in $\mathbb{R}^2$ then the line integral
$$\int f_y\,dy-f_x dx$$
   is independent of path.
   (b) Show that for any harmonic function f in $\mathbb{R}^2$, $\langle f_x,f_y\rangle$and $\langle f_y,-f_x\rangle$ form a pair of mutually orthogonal (i.e., perpendicular to each other) conservative vector fields.

6. (a) Sketch the curve C with parametric equations
$$x=\cos t,\quad y=\sin t,\quad z=\sin t,\quad 0\le t\le 2\pi.$$
   (b) Find $\int_C 2xe^{2y}\,dx+(2x^2e^{2y}+2y\cot z)\,dy-y^2\csc^2 z dz$.

(Answer: (a) an ellipse, (b) 0)

7. Let
$$\mathbf{F}(x,y)=\frac{1}{x^2+y^2}\left[(2x^3+2xy^2-2y)\mathbf{i}+(2y^3+2x^2y+2x)\mathbf{j}\right].$$
Evaluate $\oint_C \mathbf{F}\cdot d\mathbf{r}$, where C is an arbitrary positively oriented simple closed curve containing the origin in its interior.

(Answer: 4π)

8. Find the positively oriented simple closed curve C for which the value of the line integral
$$\int_C(y^3-y)\,dx-2x^3\,dy$$

is a maximum.

9. (a) As part of the lecture on div and curl, we reformulated Green's theorem as follows:

$$\oint_C \mathbf{F} \cdot d\mathbf{r} = \iint_D (\text{curl } \mathbf{F}) \cdot \mathbf{k} \, dA, \tag{1}$$

where C and D satisfy the hypotheses of Green's theorem. This led to a discussion prompted by two questions of Craig and Evan, about the significance of $\mathbf{k}$ in this formula, and possible generalizations of this result for curves C not necessarily lying in the (x, y) plane. We are now in a position to address this question in its entirety.

Let C be a simple closed curve lying on a (not necessarily horizontal) plane P, and enclosing a domain D. Let $\mathbf{F}$ be a vector field in $\mathbb{R}^3$ with continuous partial derivatives on D. Find an identity similar to (1) that relates the line integral $\oint_C \mathbf{F} \cdot d\mathbf{r}$ with an integral over the domain D. What has $\mathbf{k} \, dA$ been replaced by?

(b) Let C be a simple positively oriented closed curve lying in a plane with unit normal vector $\mathbf{n} = \langle a, b, c \rangle$. Show that the plane area enclosed by C is

$$\frac{1}{2} \oint_C (bz - cy)dx + (cx - az)dy + (ay - bx)dz.$$