Math 217, Fall 2009
Midterm 2 Solution

Name: **SID:**

Instructor: Pramanik   **Section: 101**

Instructions

- The total time is 50 minutes.
- The total score is 100 points.
- Use the reverse side of each page if you need extra space.
- Show all your work. A correct answer without intermediate steps will receive no credit.
- Calculators and cheat sheets are not allowed.

Problem	Points	Score
1	10	
2	20	
3	20	
4	20	
5	10	
6	20	
TOTAL	100	

1. Describe and sketch the solid described by the inequalities

$$\rho \leq 1, \quad \frac{3\pi}{4} \leq \phi \leq \pi, \quad 0 \leq \theta \leq 2\pi,$$

where (ρ, θ, ϕ) denote the spherical coordinates. Label and shade your diagram clearly.

(10 points)

Solution. The set $\rho \leq 1$ is the solid sphere of radius 1 centered at the origin. The set $\frac{3\pi}{4} \leq \phi \leq \pi$ is the lower half of the solid right circular cone, whose cross section at height -1 is the circle of radius 1. The solid described by both inequalities is therefore the region contained inside both the sphere and the cone, in other words the cone with vertex at the origin and height 1 pointing downwards with a spherical cap. □

2. Find the points on the surface $xy^2z^3 = 6\sqrt{3}$ closest to the origin. For full credit, you should give clear reasons why the solution you have obtained gives the shortest distance.

(20 points)

Solution. We need to minimize the function $f(x, y, z) = x^2 + y^2 + z^2$ subject to the constraint $g(x, y, z) = xy^2z^3 - 6\sqrt{3}$. Thus we have to solve the system of equations

(1) $$2x = \lambda y^2 z^3,$$

(2) $$2y = 2\lambda xyz^3,$$

(3) $$2z = 3\lambda xy^2 z^2.$$

Combining (1) and (2) we obtain $y^2 = 2x^2$, whereas combining (2) and (3) yields $z^2 = 3x^2$. Therefore $xy^2z^3 = 6\sqrt{3}x^6 = 6\sqrt{3}$, therefore $x = \pm 1$. The solutions are $(1, \pm\sqrt{2}, \sqrt{3})$ and $(-1, \pm\sqrt{2}, -\sqrt{3})$. □

3. Find the volume of the solid bounded by the cylinder $x^2 + y^2 = 4$ and the planes $z = 0$ and $y + z = 3$. Your solution should contain an integral expression for the volume, a description (verbal

or diagrammatic) of the domain of integration, and an evaluation of the integral.

(20 points)

Solution. The domain of integration is the disk $x^2 + y^2 \leq 4$. The volume equals

$$\int_{x^2+y^2\leq4} \int_{z=0}^{3-y} dV = \int_{\theta=0}^{2\pi} \int_{r=0}^{2} \int_{z=0}^{3-r\sin\theta} dz\, r\, dr\, d\theta$$

$$= \int_{\theta=0}^{2\pi} \int_{r=0}^{2} (3 - r\sin\theta)r\, dr\, d\theta$$

$$= 12\pi.$$

$\square$

4. There are two random variables X and Y whose joint probability density function f obeys the following two properties:
   (i) $f(x,y)$ vanishes outside the disk $x^2 + y^2 \leq 1$.
   (ii) For a point (x,y) in the disk, $f(x,y)$ is inversely proportional to the distance of (x,y) from the origin.
   Find $P(0 \leq X \leq Y \leq \sqrt{3}X)$.

(20 points)

Solution. The probability density function f is of the form

$$f(x,y) = \begin{cases} \dfrac{k}{\sqrt{x^2+y^2}} & \text{if } x^2 + y^2 \leq 1, \\ 0 & \text{otherwise.} \end{cases}$$

We first estimate k using the fact that the integral of f is 1:

$$1 = k \iint \frac{dy\, dx}{\sqrt{x^2 + y^2}} = k \int_{\theta=0}^{2\pi} \int_{r=0}^{1} \frac{r\, dr\, d\theta}{r} = 2\pi k, \quad \text{i.e.,} \quad k = \frac{1}{2\pi}.$$

The probability is now computed as follows,

$$P(0 \leq X \leq Y \leq \sqrt{3}X) = \frac{1}{2\pi} \int_{0}^{1} \int_{\theta=\frac{\pi}{4}}^{\frac{\pi}{3}} \frac{r\, dr\, d\theta}{r} = \frac{1}{2\pi}\frac{\pi}{12} = \frac{1}{24}.$$

$\square$

5. Evaluate

$$\int_0^{\sqrt{\pi}} \int_y^{\sqrt{\pi}} \cos(x^2)\, dx\, dy.$$

(10 points)

Solution. Interchanging the order of integration, the integral above reduces to

$$\int_{x=0}^{\sqrt{\pi}} \int_{y=0}^{x} \cos(x^2)\, dy\, dx = \frac{1}{2}\int_0^{\sqrt{\pi}} 2x\cos(x^2)\, dx = \frac{1}{2}\sin(x^2)\Big|_0^{\sqrt{\pi}} = 0.$$

□

6. Find the direction in which the function

$$f(x, y, z) = ze^{xy}$$

increases most rapidly at $(0, 0, 1)$. Describe the direction(s) in which f undergoes no change at all and the direction(s) in which f increases only half as rapidly as its maximum possible rate of increase.

$(6 + 7 + 7 = 20$ points)

Solution. The function f increases most rapidly in the direction of $\nabla f(0, 0, 1) = \langle zye^{xy}, zxe^{xy}, e^{xy}\rangle = \langle 0, 0, 1\rangle = \mathbf{k}$. It does not change along directions $\mathbf{u}$ that are perpendicular to $\mathbf{k}$, i.e., directions lying in the horizontal plane (of the form $\mathbf{u} = (\cos, \theta, \sin\theta, 0)$, $0 \le \theta \le 2\pi$). It increases half as rapidly as its maximum possible value along directions $\mathbf{u}$ whose angle θ with $\mathbf{k}$ is $\frac{\pi}{3}$. These directions lie along a cone making an angle $\frac{\pi}{3}$ with the vertical, and are of the form $\mathbf{u} = \langle \frac{\sqrt{3}}{2}\cos\theta, \frac{\sqrt{3}}{2}\sin\theta, \frac{1}{2}\rangle$, $0 \le \theta \le 2\pi$. □