Math 217, Fall 2009

Solutions to Midterm I

1. Find an equation of the plane that passes through the point $(1, 2, 3)$ and contains the line $x = 3t$, $y = 1 + t$, $z = 2 - t$.

(15 points)

Solution: The vectors $\mathbf{v}_1 = 3\mathbf{i} + \mathbf{j} - \mathbf{k}$ (which is the direction vector of the line) and $\mathbf{v}_2 = \mathbf{i} + \mathbf{j} + \mathbf{k}$ (which is the vector connecting the points $(1, 2, 3)$ and $(0, 1, 2)$) lie on the plane. Therefore the normal vector to the plane is

$$\mathbf{n} = \mathbf{v}_1 \times \mathbf{v}_2 = 2\mathbf{i} - 4\mathbf{j} + 2\mathbf{k}.$$

The equation of the plane is

$$(x - 1) - 2(y - 2) + (z - 3) = 0 \qquad \text{or} \qquad x - 2y + z = 8.$$

□

2. Identify and sketch the graph of the surface $4x^2 + 4y^2 - 8y + z^2 = 0$. Also draw a contour map with several level curves.

(5+5+5=15 points)

Solution: Rewriting the equation, we obtain $x^2 + (y - 1)^2 + \frac{z^2}{4} = 1$, which is an ellipsoid centered at $(0, 1, 0)$, with axes parallel to the coordinate axes, and axial lenths 2, 2 and 4 along the x, y and z axes respectively.

The level curves are given by $x^2 + (y - 1)^2 = 1 - k^2/4$, $|k| \leq 2$. These are concentric circles centered at $(0, 1)$, shrinking as $|k|$ increases. □

3. Find the length of the curve

$$\mathbf{r}(t) = \langle t^{\frac{3}{2}}, \cos(2t), \sin(2t)\rangle, \quad 0 \leq t \leq 1.$$

(15 points)

Solution: Since $\mathbf{r}'(t) = \langle \frac{3}{2}\sqrt{t}, -2\sin(2t), 2\cos(2t)\rangle$, therefore $|\mathbf{r}'(t)| = \sqrt{4 + \frac{9}{4}t}$. The arclength is

$$\int_0^1 \sqrt{4 + \frac{9}{4}t}\, dt = \frac{61}{27}.$$

□

4. Two particles move along paths given by

$$\mathbf{r}_1(t) = \langle t^2, 7t - 12, t^2\rangle, \qquad \mathbf{r}_2(t) = \langle 4t - 3, t^2, 5t - 6\rangle$$

where t denotes time in seconds. Show that the two particles collide, and determine the time of collision.

(10 points)

Solution: The two particles will collide if there is a time t for which all three equations $t^2 = 4t - 3$, $7t - 12 = t^2$, $t^2 = 5t - 6$ are satisfied. We observe that $t = 3$ is a solution for all three equations. □

5. Find the equation of the tangent plane to the surface

$$z = y\cos(x - y)$$

at $(2, 2, 2)$.

(15 points)

Solution: We compute at $(2,2,2)$

$$\frac{\partial z}{\partial x} = -y\sin(x-y) = 0, \qquad \frac{\partial z}{\partial y} = \cos(x-y) + y\sin(x-y) = 1.$$

The equation of the tangent plane is

$$z - 2 = 0(x-2) + 1(y-2) \qquad \text{or } z = y.$$

□

6. The radius of a right circular cone is increasing at the rate of 1 in/s while its height is decreasing at a rate of 2 in/s. At what rate is the volume of the cone changing when the radius is 10 in and the height is 5 in.? (Hint: Recall that the volume of a right circular cone is $\frac{\pi}{3}$ times the square of the radius times the height.)

(15 points)

Solution: Since $V = \frac{1}{3}\pi r^2 h$, we obtain by differentiating with respect to t,

$$\frac{dV}{dt} = \frac{\pi}{3}\left[2r\frac{dr}{dt}h + r^2\frac{dh}{dt} = \frac{\pi}{3}\left((20)(1)(5) - (100)(2)\right)\right] = -\frac{100\pi}{3}\text{in}^3/\text{s}.$$

□

7. Determine whether the following limit exists. If yes, evaluate the limit. If the limit does not exist, explain why.

$$\lim_{(x,y)\to(0,0)} \frac{x^2 + \sin^2 y}{2x^2 + y^2}.$$

Solution: Choose a linear path of the form $y = mx$ converging to $(0,0)$. Then

$$\frac{x^2 + \sin^2 y}{2x^2 + y^2} = \frac{x^2 + \sin^2(mx)}{2x^2 + (mx)^2} = \frac{1 + \frac{\sin^2(mx)}{x^2}}{2 + m^2}.$$

Since $\lim_{x\to 0} \sin(mx)/x = m$ by L'Hopital's rule, we obtain

$$\lim_{x\to 0} \frac{1 + \frac{\sin^2(mx)}{x^2}}{2 + m^2} = \frac{1 + m^2}{2 + m^2},$$

which depends on m. Thus the limit does not exist. □