12. $\mathbf{F}(x,y) = \langle y^2\cos x, x^2 + 2y\sin x\rangle$ and the region D enclosed by C is given by $\{(x,y) \mid 0 \le x \le 2, 0 \le y \le 3x\}$.

C is traversed clockwise, so $-C$ gives the positive orientation.

$$\begin{aligned}\int_C \mathbf{F}\cdot d\mathbf{r} &= -\int_{-C}(y^2\cos x)\,dx + (x^2 + 2y\sin x)\,dy = -\iint_D \left[\tfrac{\partial}{\partial x}(x^2 + 2y\sin x) - \tfrac{\partial}{\partial y}(y^2\cos x)\right] dA \\ &= -\iint_D (2x + 2y\cos x - 2y\cos x)\,dA = -\int_0^2\int_0^{3x} 2x\,dy\,dx \\ &= -\int_0^2 2x\,\big[y\big]_{y=0}^{y=3x}\,dx = -\int_0^2 6x^2\,dx = -2x^3\big]_0^2 = -16\end{aligned}$$

19. Let C_1 be the arch of the cycloid from $(0,0)$ to $(2\pi,0)$, which corresponds to $0 \le t \le 2\pi$, and let C_2 be the segment from $(2\pi,0)$ to $(0,0)$, so C_2 is given by $x = 2\pi - t$, $y = 0$, $0 \le t \le 2\pi$. Then $C = C_1 \cup C_2$ is traversed clockwise, so $-C$ is oriented positively. Thus $-C$ encloses the area under one arch of the cycloid and from (5) we have

$$\begin{aligned}A = -\oint_{-C} y\,dx &= \int_{C_1} y\,dx + \int_{C_2} y\,dx = \int_0^{2\pi}(1-\cos t)(1-\cos t)\,dt + \int_0^{2\pi} 0\,(-dt) \\ &= \int_0^{2\pi}(1 - 2\cos t + \cos^2 t)\,dt + 0 = \left[t - 2\sin t + \tfrac{1}{2}t + \tfrac{1}{4}\sin 2t\right]_0^{2\pi} = 3\pi\end{aligned}$$

22. By Green's Theorem, $\frac{1}{2A}\oint_C x^2\,dy = \frac{1}{2A}\iint_D 2x\,dA = \frac{1}{A}\iint_D x\,dA = \overline{x}$ and $-\frac{1}{2A}\oint_C y^2\,dx = -\frac{1}{2A}\iint_D (-2y)\,dA = \frac{1}{A}\iint_D y\,dA = \overline{y}$.

16. $\operatorname{curl}\mathbf{F} = \nabla\times\mathbf{F} = \begin{vmatrix}\mathbf{i} & \mathbf{j} & \mathbf{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ e^z & 1 & xe^z\end{vmatrix} = (0-0)\,\mathbf{i} - (e^z - e^z)\,\mathbf{j} + (0-0)\,\mathbf{k} = \mathbf{0}$ and $\mathbf{F}$ is defined on all of $\mathbb{R}^3$ with component functions that have continuous partial deriatives, so $\mathbf{F}$ is conservative. Thus there exists a function f such that $\nabla f = \mathbf{F}$. Then $f_x(x,y,z) = e^z$ implies $f(x,y,z) = xe^z + g(y,z) \quad\Rightarrow\quad f_y(x,y,z) = g_y(y,z)$. But $f_y(x,y,z) = 1$, so $g(y,z) = y + h(z)$ and $f(x,y,z) = xe^z + y + h(z)$. Thus $f_z(x,y,z) = xe^z + h'(z)$ but $f_z(x,y,z) = xe^z$, so $h(z) = K$, a constant. Hence a potential function for $\mathbf{F}$ is $f(x,y,z) = xe^z + y + K$.

20. No. Assume there is such a $\mathbf{G}$. Then $\operatorname{div}(\operatorname{curl}\mathbf{G}) = yz - 2yz + 2yz = yz \ne 0$ which contradicts Theorem 11.

32. $\mathbf{r} = x\,\mathbf{i} + y\,\mathbf{j} + z\,\mathbf{k} \quad\Rightarrow\quad r = |\mathbf{r}| = \sqrt{x^2+y^2+z^2}$, so

$$\mathbf{F} = \frac{\mathbf{r}}{r^p} = \frac{x}{(x^2+y^2+z^2)^{p/2}}\,\mathbf{i} + \frac{y}{(x^2+y^2+z^2)^{p/2}}\,\mathbf{j} + \frac{z}{(x^2+y^2+z^2)^{p/2}}\,\mathbf{k}$$

Then $\dfrac{\partial}{\partial x}\dfrac{x}{(x^2+y^2+z^2)^{p/2}} = \dfrac{(x^2+y^2+z^2) - px^2}{(x^2+y^2+z^2)^{1+p/2}} = \dfrac{r^2 - px^2}{r^{p+2}}$. Similarly,

$\dfrac{\partial}{\partial y}\dfrac{y}{(x^2+y^2+z^2)^{p/2}} = \dfrac{r^2 - py^2}{r^{p+2}}$ and $\dfrac{\partial}{\partial z}\dfrac{z}{(x^2+y^2+z^2)^{p/2}} = \dfrac{r^2 - pz^2}{r^{p+2}}$. Thus

$$\begin{aligned}\operatorname{div}\mathbf{F} = \nabla\cdot\mathbf{F} &= \frac{r^2 - px^2}{r^{p+2}} + \frac{r^2 - py^2}{r^{p+2}} + \frac{r^2 - pz^2}{r^{p+2}} = \frac{3r^2 - px^2 - py^2 - pz^2}{r^{p+2}} \\ &= \frac{3r^2 - p(x^2+y^2+z^2)}{r^{p+2}} = \frac{3r^2 - pr^2}{r^{p+2}} = \frac{3-p}{r^p}\end{aligned}$$

Consequently, if $p = 3$ we have $\operatorname{div}\mathbf{F} = 0$.

33. By (13), $\oint_C f(\nabla g)\cdot \mathbf{n}\,ds = \iint_D \operatorname{div}(f\nabla g)\,dA = \iint_D [f\operatorname{div}(\nabla g) + \nabla g\cdot\nabla f]\,dA$ by Exercise 25. But $\operatorname{div}(\nabla g) = \nabla^2 g$. Hence $\iint_D f\nabla^2 g\,dA = \oint_C f(\nabla g)\cdot\mathbf{n}\,ds - \iint_D \nabla g\cdot\nabla f\,dA$.

4. $\mathbf{r}(u,v) = 2\sin u\,\mathbf{i} + 3\cos u\,\mathbf{j} + v\,\mathbf{k}$, so the corresponding parametric equations for the surface are $x = 2\sin u$, $y = 3\cos u$, $z = v$. For any point (x,y,z) on the surface, we have $(x/2)^2 + (y/3)^2 = \sin^2 u + \cos^2 u = 1$, so cross-sections parallel to the yz-plane are all ellipses. Since $z = v$ with $0 \le v \le 2$, the surface is the portion of the elliptical cylinder $x^2/4 + y^2/9 = 1$ for $0 \le z \le 2$.

6. $\mathbf{r}(s,t) = s\sin 2t\,\mathbf{i} + s^2\,\mathbf{j} + s\cos 2t\,\mathbf{k}$, so the corresponding parametric equations for the surface are $x = s\sin 2t$, $y = s^2$, $z = s\cos 2t$. For any point (x,y,z) on the surface, we have $x^2 + z^2 = s^2\sin^2 2t + s^2\cos^2 2t = s^2 = y$. Since no restrictions are placed on the parameters, the surface is $y = x^2 + z^2$, which we recognize as a circular paraboloid whose axis is the y-axis.

42. $z = f(x,y) = 1 + 3x + 2y^2$ with $0 \le x \le 2y$, $0 \le y \le 1$. Thus, by Formula 9,

$$A(S) = \iint_D \sqrt{1 + 3^2 + (4y)^2}\,dA = \int_0^1 \int_0^{2y} \sqrt{10 + 16y^2}\,dx\,dy = \int_0^1 2y\sqrt{10 + 16y^2}\,dy$$

$$= \tfrac{1}{16}\cdot\tfrac{2}{3}(10 + 16y^2)^{3/2}\Big]_0^1 = \tfrac{1}{24}(26^{3/2} - 10^{3/2})$$

44. A parametric representation of the surface is $x = y^2 + z^2$, $y = y$, $z = z$ with $0 \le y^2 + z^2 \le 9$.

Hence $\mathbf{r}_y \times \mathbf{r}_z = (2y\,\mathbf{i} + \mathbf{j}) \times (2z\,\mathbf{i} + \mathbf{k}) = \mathbf{i} - 2y\,\mathbf{j} - 2z\,\mathbf{k}$.

Note: In general, if $x = f(y,z)$ then $\mathbf{r}_y \times \mathbf{r}_z = \mathbf{i} - \dfrac{\partial f}{\partial y}\mathbf{j} - \dfrac{\partial f}{\partial z}\mathbf{k}$, and $A(S) = \displaystyle\iint_D \sqrt{1 + \left(\frac{\partial f}{\partial y}\right)^2 + \left(\frac{\partial f}{\partial z}\right)^2}\,dA$. Then

$$A(S) = \iint_{0 \le y^2 + z^2 \le 9} \sqrt{1 + 4y^2 + 4z^2}\,dA = \int_0^{2\pi}\int_0^3 \sqrt{1 + 4r^2}\,r\,dr\,d\theta$$

$$= \int_0^{2\pi} d\theta \int_0^3 r\sqrt{1 + 4r^2}\,dr = 2\pi\left[\tfrac{1}{12}(1 + 4r^2)^{3/2}\right]_0^3 = \tfrac{\pi}{6}(37\sqrt{37} - 1)$$