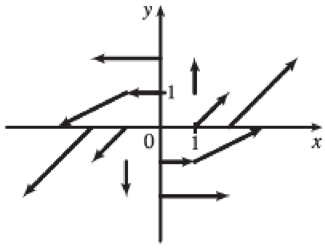
4. $\mathbf{F}(x, y) = (x - y)\,\mathbf{i} + x\,\mathbf{j}$

The length of the vector $(x - y)\,\mathbf{i} + x\,\mathbf{j}$ is $\sqrt{(x-y)^2 + x^2}$. Vectors along the line $y = x$ are vertical.

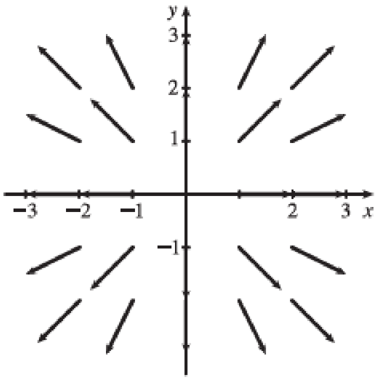


26. $f(x, y) = \sqrt{x^2 + y^2} \quad \Rightarrow$

$$\nabla f(x, y) = \tfrac{1}{2}(x^2 + y^2)^{-1/2}(2x)\,\mathbf{i} + \tfrac{1}{2}(x^2 + y^2)^{-1/2}(2y)\,\mathbf{j}$$

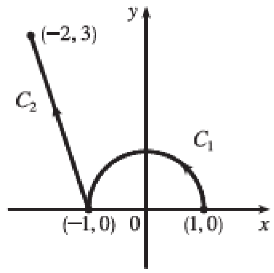
$$= \frac{x}{\sqrt{x^2 + y^2}}\,\mathbf{i} + \frac{y}{\sqrt{x^2 + y^2}}\,\mathbf{j} \text{ or } \frac{1}{\sqrt{x^2 + y^2}}\,(x\,\mathbf{i} + y\,\mathbf{j})\,.$$

$\nabla f(x, y)$ is not defined at the origin, but elsewhere all vectors have length 1 and point away from the origin.



34. At $t = 1$ the particle is at $(1, 3)$ so its velocity is $\mathbf{F}(1, 3) = \langle 1, -1 \rangle$. After 0.05 units of time, the particle's change in location should be approximately $0.05\,\mathbf{F}(1, 3) = 0.05\,\langle 1, -1 \rangle = \langle 0.05, -0.05 \rangle$, so the particle should be approximately at the point $(1.05, 2.95)$.

8.



$C = C_1 + C_2$

On C_1: $x = \cos t \;\Rightarrow\; dx = -\sin t\,dt,\;\; y = \sin t \;\Rightarrow\; dy = \cos t\;dt,\;\; 0 \le t \le \pi.$

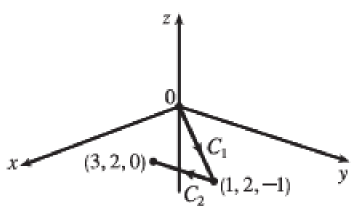
On C_2: $x = -1 - t \;\Rightarrow\; dx = -dt,\; y = 3t \;\Rightarrow\; dy = 3\,dt,\;\; 0 \le t \le 1.$

Then

$$\begin{aligned}
\int_C \sin x\,dx + \cos y\,dy &= \int_{C_1} \sin x\,dx + \cos y\,dy + \int_{C_2} \sin x\,dx + \cos y\,dy \\
&= \int_0^\pi \sin(\cos t)(-\sin t\,dt) + \cos(\sin t)\cos t\,dt + \int_0^1 \sin(-1-t)(-dt) + \cos(3t)(3\,dt) \\
&= \big[-\cos(\cos t) + \sin(\sin t)\big]_0^\pi + \big[-\cos(-1-t) + \sin(3t)\big]_0^1 \\
&= -\cos(\cos\pi) + \sin(\sin\pi) + \cos(\cos 0) - \sin(\sin 0) - \cos(-2) + \sin(3) + \cos(-1) - \sin(0) \\
&= -\cos(-1) + \sin 0 + \cos(1) - \sin 0 - \cos(-2) + \sin 3 + \cos(-1) \\
&= -\cos 1 + \cos 1 - \cos 2 + \sin 3 + \cos 1 = \cos 1 - \cos 2 + \sin 3
\end{aligned}$$

where we have used the identity $\cos(-\theta) = \cos\theta$.

16.



On C_1: $x = t \;\Rightarrow\; dx = dt,\; y = 2t \;\Rightarrow$

$dy = 2\,dt,\; z = -t \;\Rightarrow\; dz = -dt,\; 0 \le t \le 1.$

On C_2: $= 1 + 2t \;\Rightarrow\; dx = 2\,dt,\; y = 2 \;\Rightarrow$

$dy = 0\,dt,\; z = -1 + t \;\Rightarrow\; dz = dt,\; 0 \le t \le 1.$

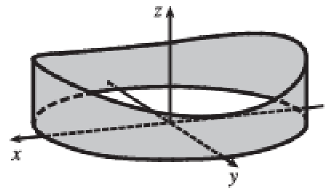
Then

$$\begin{aligned}
\int_C x^2\,dx + y^2\,dy + z^2\,dz &= \int_{C_1} x^2\,dx + y^2\,dy + z^2\,dz + \int_{C_2} x^2\,dx + y^2\,dy + z^2\,dz \\
&= \int_0^1 t^2\,dt + (2t)^2 \cdot 2\,dt + (-t)^2(-dt) + \int_0^1 (1+2t)^2 \cdot 2\,dt + 2^2 \cdot 0\,dt + (-1+t)^2\,dt \\
&= \int_0^1 8t^2\,dt + \int_0^1 (9t^2 + 6t + 3)\,dt = \left[\tfrac{8}{3}t^3\right]_0^1 + \left[3t^3 + 3t^2 + 3t\right]_0^1 = \tfrac{35}{3}
\end{aligned}$$

40. $x = x,\;\; y = x^2,\;\; -1 \le x \le 2,$

$$W = \int_{-1}^2 \langle x\sin x^2, x^2\rangle \cdot \langle 1, 2x\rangle\,dx = \int_{-1}^2 (x\sin x^2 + 2x^3)\,dx = \left[-\tfrac{1}{2}\cos x^2 + \tfrac{1}{2}x^4\right]_{-1}^2 = \tfrac{1}{2}(15 + \cos 1 - \cos 4).$$

46. Consider the base of the fence in the xy-plane, centered at the origin, with the height given by $z = h(x, y)$. The fence can be graphed using the parametric equations $x = 10\cos u$, $y = 10\sin u$,



$$z = v\left[4 + 0.01((10\cos u)^2 - (10\sin u)^2)\right] = v(4 + \cos^2 u - \sin^2 u)$$
$$= v(4 + \cos 2u),\ 0 \le u \le 2\pi,\ 0 \le v \le 1.$$

The area of the fence is $\int_C h(x, y)\,ds$ where C, the base of the fence, is given by $x = 10\cos t$, $y = 10\sin t$, $0 \le t \le 2\pi$. Then

$$\int_C h(x,y)\,ds = \int_0^{2\pi} \left[4 + 0.01((10\cos t)^2 - (10\sin t)^2)\right] \sqrt{(-10\sin t)^2 + (10\cos t)^2}\,dt$$
$$= \int_0^{2\pi} (4 + \cos 2t)\sqrt{100}\,dt = 10\left[4t + \tfrac{1}{2}\sin 2t\right]_0^{2\pi} = 10(8\pi) = 80\pi \text{ m}^2$$

If we paint both sides of the fence, the total surface area to cover is 160π m^2, and since 1 L of paint covers 100 m^2, we require $\frac{160\pi}{100} = 1.6\pi \approx 5.03$ L of paint.

48. Use the orientation pictured in the figure. Then since $\mathbf{B}$ is tangent to any circle that lies in the plane perpendicular to the wire, $\mathbf{B} = |\mathbf{B}|\,\mathbf{T}$ where $\mathbf{T}$ is the unit tangent to the circle C: $x = r\cos\theta$, $y = r\sin\theta$. Thus $\mathbf{B} = |\mathbf{B}|\,\langle -\sin\theta, \cos\theta\rangle$. Then $\int_C \mathbf{B}\cdot d\mathbf{r} = \int_0^{2\pi} |\mathbf{B}|\,\langle -\sin\theta, \cos\theta\rangle \cdot \langle -r\sin\theta, r\cos\theta\rangle\,d\theta = \int_0^{2\pi} |\mathbf{B}|\,r\,d\theta = 2\pi r\,|\mathbf{B}|$. (Note that $|\mathbf{B}|$ here is the magnitude of the field at a distance r from the wire's center.) But by Ampere's Law $\int_C \mathbf{B}\cdot d\mathbf{r} = \mu_0 I$. Hence $|\mathbf{B}| = \mu_0 I/(2\pi r)$.

16. (a) $f_x(x,y,z) = 2xz + y^2$ implies $f(x,y,z) = x^2 z + xy^2 + g(y,z)$ and so $f_y(x,y,z) = 2xy + g_y(y,z)$. But $f_y(x,y,z) = 2xy$ so $g_y(y,z) = 0 \quad\Rightarrow\quad g(y,z) = h(z)$. Thus $f(x,y,z) = x^2 z + xy^2 + h(z)$ and $f_z(x,y,z) = x^2 + h'(z)$. But $f_z(x,y,z) = x^2 + 3z^2$, so $h'(z) = 3z^2 \quad\Rightarrow\quad h(z) = z^3 + K$. Hence $f(x,y,z) = x^2 z + xy^2 + z^3$ (taking $K = 0$).

(b) $t = 0$ corresponds to the point $(0,1,-1)$ and $t = 1$ corresponds to $(1,2,1)$, so $\int_C \mathbf{F}\cdot d\mathbf{r} = f(1,2,1) - f(0,1,-1) = 6 - (-1) = 7$.

26. $\nabla f(x,y) = \cos(x - 2y)\,\mathbf{i} - 2\cos(x - 2y)\,\mathbf{j}$

(a) We use Theorem 2: $\int_{C_1} \mathbf{F}\cdot d\mathbf{r} = \int_{C_1} \nabla f\cdot d\mathbf{r} = f(\mathbf{r}(b)) - f(\mathbf{r}(a))$ where C_1 starts at $t = a$ and ends at $t = b$. So because $f(0,0) = \sin 0 = 0$ and $f(\pi,\pi) = \sin(\pi - 2\pi) = 0$, one possible curve C_1 is the straight line from $(0,0)$ to (π,π); that is, $\mathbf{r}(t) = \pi t\,\mathbf{i} + \pi t\,\mathbf{j}$, $0 \le t \le 1$.

(b) From (a), $\int_{C_2} \mathbf{F}\cdot d\mathbf{r} = f(\mathbf{r}(b)) - f(\mathbf{r}(a))$. So because $f(0,0) = \sin 0 = 0$ and $f\left(\frac{\pi}{2}, 0\right) = 1$, one possible curve C_2 is $\mathbf{r}(t) = \frac{\pi}{2} t\,\mathbf{i}$, $0 \le t \le 1$, the straight line from $(0,0)$ to $\left(\frac{\pi}{2}, 0\right)$.

28. Here $\mathbf{F}(x,y,z) = y\,\mathbf{i} + x\,\mathbf{j} + xyz\,\mathbf{k}$. Then using the notation of Exercise 27, $\partial P/\partial z = 0$ while $\partial R/\partial x = yz$. Since these aren't equal, $\mathbf{F}$ is not conservative. Thus by Theorem 4, the line integral of $\mathbf{F}$ is not independent of path.

32. $D = \{(x, y) \mid x^2 + y^2 \le 1 \text{ or } 4 \le x^2 + y^2 \le 9\} =$ the points on or inside the circle $x^2 + y^2 = 1$, together with the points on or between the circles $x^2 + y^2 = 4$ and $x^2 + y^2 = 9$.

(a) D is not open because, for instance, no disk with center $(0, 2)$ lies entirely within D.

(b) D is not connected because, for example, $(0, 0)$ and $(0, 2.5)$ lie in D but cannot be joined by a path that lies entirely in D.

(c) D is not simply-connected because, for example, $x^2 + y^2 = 9$ is a simple closed curve in D but encloses points that are not in D.