Math 440/508, Fall 2008, Midterm
(Due date: Monday October 20)

Instructions

- The midterm will be collected at the end of lecture on Monday. **There will be no extensions for the midterm.**
- Unlike homework assignments, you must work on the midterm on your own. If you need hints or clarifications, please feel free to talk to the instructor.
- Answers should be clear, legible, and in complete English sentences. Solutions must be self-contained – only results proved in class can be used without proof.

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1. Suppose f is analytic in some region containing $\overline{B}(0;1)$ and $|f(z)| = 1$ where $|z| = 1$. Find a formula for f.
   Hint : First consider the case where f has no zeros in $B(0;1)$.
2. For each of the following problems, determine whether there exists a function f satisfying the prescribed set of conditions. Provide arguments in support of your answer. In the event such a function does exist, determine whether it is unique and find its form.
   (a) f is analytic in an open set containing $\overline{B}(0;1)$, $|f(z)| = 1$ when $|z| = 1$, f has a simple zero at $\frac{1}{4}(1+i)$, and a double zero at $z = \frac{1}{2}$, $f(0) = \frac{1}{2}$.
   (b) f is analytic on $B(0;1)$ such that $|f(z)| < 1$ for $|z| < 1$, $f(0) = \frac{1}{2}$, $f'(0) = \frac{3}{4}$.
3. In class, we characterized the group of conformal automorphisms (i.e., conformal self-maps) of the unit disk. This problem addresses a similar question for two other domains.
   (a) Show that any conformal self-map of the punctured complex plane $\mathbb{C} \setminus \{0\}$ is either a multiplication $z \mapsto az$ for some nonzero constant a, or such a multiplication followed by the inversion $z \mapsto \frac{1}{z}$.
   (b) Let $\Delta = \mathbb{C} \setminus \{a_1, \cdots a_m\}$ be the complex plane punctured at m distinct points. Characterize all the conformal self-maps of Δ. If possible, identify (via algebraic isomorphisms of course) the automorphism group $\mathrm{Aut}(\Delta)$ as a subgroup of a standard group you already know. Discuss the cases when this group is nontrivial and when it is not.
4. *A domain D in the extended complex plane $\mathbb{C}_\infty = \mathbb{C} \cup \{\infty\}$ is simply connected if and only if its complement $\mathbb{C}_\infty \setminus D$ is connected, or equivalently if and only if $\mathbb{C} \setminus D$ has no compact connected component.*

   Assuming this fact if necessary, prove that if a sequence of polynomials converges uniformly in a region Ω then it converges uniformly in a simply connected region containing Ω.
5. Which of the following regions are simply connected? Justify your answer.
   i. $D = \mathbb{C}_\infty \setminus [-1, 1]$,
   ii. $D = \mathbb{C}_\infty \setminus \{-1, 0, 1\}$,
   iii. $D = \mathbb{C} \setminus \{re^{ir} : 0 \leq r < \infty\}$,
   iv. $D =$ the bounded region in $\mathbb{C}$ enclosed by a smooth simple curve,
   v. $D =$ the unbounded region in $\mathbb{C}$ lying outside a smooth simple curve.
6. Find a conformal mapping that sends the common part of the disks $|z| < 1$ and $|z-1| < 1$ to the unit disk.