Homework 3 – Math 440/508

Due Friday November 7 at the end of lecture.

Problem 1. Let (Ω, d) be a metric space, $G \subseteq \mathbb{C}$ an open set, and ρ the metric we have defined in class on the space of continuous functions $C(G, \Omega)$. Prove the following facts about ρ:

(a) If $\epsilon > 0$ is given then there is a $\delta > 0$ and a compact set $K \subset G$ such that for any $f, g \in C(G, \Omega)$

$$\sup_{z \in K} d(f(z), g(z)) < \delta \implies \rho(f, g) < \epsilon.$$

(b) Conversely, if $\delta > 0$ and a compact set K are given, there is an $\epsilon > 0$ such that for $f, g \in C(g, \Omega)$,

$$\rho(f, g) < \epsilon \implies \sup_{z \in K} d(f(z), g(z)) < \delta.$$

(c) Use parts (a) and (b) to obtain the following characterization of open sets in the topology $(C(G, \Omega), \rho)$: A set $\mathcal{O} \subset C(G, \Omega)$ is open if and only if for each $f \in \mathcal{O}$ there is a compact set K and a $\delta > 0$ such that

$$\{g : d(f(z), g(z)) < \delta, z \in K\} \subset \mathcal{O}.$$

Problem 2. Recall that the definition of the metric ρ on $C(G, \Omega)$ involved a sequence $\{K_n : n \geq 1\}$ of compact subsets of G. Show that the topology on $C(G, \Omega)$ generated by ρ is invariant under the choice of these sets.

Problem 3. Show that $(C(G, \Omega), \rho)$ is a complete metric space if and only if (Ω, d) is complete.

Problem 4. Suppose that $\mathcal{F} \subseteq C(G, \Omega)$ is equicontinuous at each point of G; then show that $\mathcal{F}$ is equicontinuous over each compact subset of G.

Problem 5. Let (X_n, d_n) be a metric space for each $n \geq 1$ and let $X = \prod_{n=1}^{\infty} X_n$ be their Cartesian product. That is, $X = \{\xi = \{x_n\} : x_n \in X_n \text{ for each } n \geq 1\}$. For $\xi = \{x_n\}$ and $\eta = \{y_n\}$ in X define

$$d(\xi, \eta) = \sum_{n=1}^{\infty} 2^{-n} \frac{d_n(x_n, y_n)}{1 + d_n(x_n, y_n)}.$$

(a) Verify that (X, d) as defined above is a metric space.

(b) Show that convergence in the topology of X is equivalent to coordinatewise convergence.

(c) If (X_n, d_n) is compact for every $n \geq 1$, then X is compact.

Problem 6. This exercise shows how the mean square convergence dominates the uniform convergence of analytic functions. If U is an open subset of $\mathbb{C}$ we use the notation

$$||f||_{L^2(U)} = \left[\int_U |f(z)|^2 \, dxdy \right]^{\frac{1}{2}}$$

for the mean square norm, and

$$||f||_{L^\infty(U)} = \sup_{z \in U} |f(z)|$$

for the sup norm.

(a) If f is holomorphic in a neighborhood of the disk $B(z_0; r)$, show that for any $0 < s < r$ there exists a constant $C_{r,s}$ (depending only on r and s but independent of f or z_0) such that

$$||f||_{L^\infty(B(z_0;s))} \leq C_{r,s} ||f||_{L^2(B(z_0;r))}.$$

(b) Prove that if $\{f_n\}$ is a Cauchy sequence of holomorphic functions in the mean square norm $|| \cdot ||_{L^2(U)}$, then the sequence $\{f_n\}$ converges uniformly on every compact subset U to a holomorphic function.

Problem 7.

(a) Let X and Ω be metric spaces and suppose that $f : X \to \Omega$ is one-one and onto. Show that f is an open map if and only if f is a closed map.

(b) Let $P : \mathbb{C} \to \mathbb{R}$ be defined by $P(z) = \mathrm{Re}(z)$. Show that P is an open map but not closed.

Problem 8. It is easy to see that pointwise convergence of a sequence of continuous functions is in general a weaker condition than convergence of compacta. However, under certain additional hypotheses, the former implies the latter. This problem illustrates two such examples.

(a) Consider $C(G, \mathbb{R})$ and suppose that $\{f_n\}$ is a sequence in $C(G, \mathbb{R})$ which is monotonically increasing (i.e., $f_n(z) \leq f_{n+1}(z)$ for all $z \in G$) and satisfies

$$\lim_{n \to \infty} f_n(z) = f(z) \quad \text{for all } z \in G \text{ where } f \in C(G, \mathbb{R}).$$

Show that $f_n \xrightarrow{\rho} f$. This result is due to Dini.

(b) Let $\{f_n\} \subset C(G, \Omega)$ and suppose that $\{f_n\}$ is equicontinuous. If $f \in C(G, \Omega)$ and $f_n(z) \to f(z)$ for each z then show that $f_n \xrightarrow{\rho} f$.

Problem 9. Let f be analytic on the unit disk with series expansion $\sum_{n \geq 0} a_n z^n$ at 0. Let $\mu = \inf\{|f(z)| : |z| = 1\}$, and suppose that f has at most m zeroes in $B(0; 1)$. Prove that

$$\mu \leq |a_0| + \cdots + |a_m|.$$

Problem 10. Let $\mathcal{F} \subseteq C(G, \mathbb{C})$, where G is an open subset of $\mathbb{C}$. If $\mathcal{F}$ is pointwise bounded, i.e.,

$$\sup\{|f(z)| : f \in \mathcal{F}\} < \infty \qquad \text{for all } z \in G,$$

does this imply that $\mathcal{F}$ is locally bounded? Would your answer change if $\mathcal{F}$ was required to be a subset of $\mathbb{H}(G)$?
(*Hint :* For the second part of the question, argue that it suffices to create a sequence of analytic functions that converges pointwise to a non-analytic function. Then construct such a sequence of analytic functions (in fact a sequence of polynomials), taking for granted the following fact: *Let K be a compact subset of the complex plane with connected complement. Let f be a function analytic in a neighborhood of K. Then there is a sequence of polynomials which converges to f uniformly on K.* Choose a sequence of compact (but not necessarily connected) sets K_n gradually filling up $\mathbb{C}$, and a sequence of functions g_n analytic on an open neighborhood of K_n on which the statement above may be applied.)