Homework 1 – Math 440/508

Due Friday September 19 at the end of lecture

Problem 1. Show the following:

a. If $\mathrm{Im}(a) > 0$, $\mathrm{Im}(b) < 0$ and $b \neq a$,

$$\frac{\mathrm{Im}(a)}{\pi} \int_{-\infty}^{\infty} \frac{\log|x-b|}{|x-a|^2}\, dx = \log|b-a|.$$

b.

$$\int_0^{\infty} \frac{\sin(x^2)}{x}\, dx = \frac{\pi}{4}.$$

c.

$$\int_0^{\infty} \frac{\cos x}{\sqrt{x}}\, dx = \sqrt{\frac{\pi}{2}}.$$

d.

$$\int_0^{\infty} \frac{x^{a-1}}{1+x+x^2}\, dx = \frac{2\pi}{\sqrt{3}} \frac{\cos\left(\frac{\pi}{3}a + \frac{\pi}{6}\right)}{\sin(\pi a)}, \quad 0 < a < 2.$$

e.

$$\int_0^{\infty} \frac{(\log x)^2}{1+x+x^2}\, dx = \frac{16}{81} \frac{\pi^3}{\sqrt{3}}.$$

Hint: Try computing $\int_0^{\infty} (1+x+x^2)^{-1} \log x\, dx$ first.

f.

$$\int_0^{2\pi} \log \sin^2(2\theta)\, d\theta = 4 \int_0^{\pi} \log \sin\theta\, d\theta = -4\pi \log 2.$$

Hint: Take the function $f(z) = \log(1 - e^{2iz})$ as the integrand and use a suitable contour.

g.

$$\int_0^{\infty} \frac{\sin(ax)}{e^{2\pi x} - 1}\, dx = \frac{1+e^{-a}}{4(1-e^{-a})} - \frac{1}{2a}$$

Hint: Use a rectangular contour, conveniently indented.

Problem 2. Let f be an entire function and let $a, b \in \mathbb{C}$ be such that $|a| < R$, $|b| < R$. Evaluate the following integral:

$$\int_{|z|=R} \frac{f(z)}{(z-a)(z-b)}\, dz.$$

Use this result to give another proof of Liouville's theorem (the only bounded entire functions are the constants).