Math 263 Fall 2008, Test 2 Solutions

1. Let $\mathbf{F}(x, y, z) = (\sin x, 2\cos x, 1 - y^2)$.
   (a) Calculate curl $\mathbf{F}$.
   (b) Calculate div $\mathbf{F}$.
   (c) Calculate div(curl $\mathbf{F}$).
   Solution:
   (a) curl $\mathbf{F} = (-2y)\vec{i} - (2\sin x)\vec{k}$.
   (b) div $\mathbf{F} = \cos x$.
   (c) div(curl $\mathbf{F}$)=0. This is true for any vector field.

2. Sketch the domain of integration for the integral given below. Then convert the integral to spherical coordinates and evaluate it.

$$\int_{-3}^{3} \int_{-\sqrt{9-x^2}}^{\sqrt{9-x^2}} \int_{0}^{\sqrt{9-x^2-y^2}} z\sqrt{x^2+y^2+z^2} \quad dzdydx$$

   Solution:
   The integral represents the top half of a sphere of radius 3, centred at the origin. Converting to spherical coordinates, we get:

$$\begin{aligned} &\int_0^{2\pi} \int_0^{\pi/2} \int_0^3 (\rho\cos\phi)\rho \quad \rho^2 \sin\phi \quad d\rho d\phi d\theta \\ =&2\pi \left(\int_0^3 \rho^4 d\rho\right)\left(\int_0^{\pi/2} \cos\phi\sin\phi d\phi\right) \\ =&2\pi\left(\frac{3^5}{5}\right)\int_0^{\pi/2} \frac{1}{2}\sin 2\phi d\phi \\ =&2\pi\left(\frac{3^5}{5}\right)\left[-\frac{1}{4}\cos 2\phi\right]_0^{\pi/2} \\ =&2\pi\left(\frac{3^5}{5}\right)\left(\frac{1}{4}+\frac{1}{4}\right) = \frac{243\pi}{5} \end{aligned}$$

3. Is the vector field $\mathbf{F}(x, y, z) = (2xy + y^2)\mathbf{i} + (x^2 + 2xy + z^2)\mathbf{j} + 2zy\mathbf{k}$ conservative? If so, find a function f so that $\mathbf{F} = \nabla f$. If not, explain clearly why.
   Solution:
   We can check that F is conservative by checking that the curl is zero. Alternatively, you could just try to create a function $f(x, y, z)$ so that $\mathbf{F} = \nabla f$. By partially integrating $(2xy + y^2)$ with respect to x, you get $f = x^2y + y^2x + ...$ By partially integrating $(x^2 + 2xy + z^2)$ with respect to y, you get $f = x^2y + xy^2 + z^2y + ...$ By partially integrating $2zy$ with respect to z, you get $f = z^2y + ...$ Putting these all together, $f = x^2y + y^2x + z^2y$ is a potential function for the vector field, and the vector field is therefore conservative.

4. Find the line integral of $\mathbf{F}(x, y, z) = (yz)\mathbf{i} + (xz)\mathbf{j} + (xy + 1)\mathbf{k}$ around the square with corners at $(0, 0, 1), (1, 0, 1), (1, 1, 1)$ and $(0, 1, 1)$ (taken in that order).
   Solution:
   This is a vector integral around a closed curve in 3-d. Green's theorem does not apply because it is in 3-d. However, if $\mathbf{F}$ is conservative, then we know that the integral will be

0. We can check if F is conservative using the curl test: curl$\mathbf{F} = 0$ (check it!). Therefore F is conservative and the integral of F around any closed curve is zero.

5. (a) State Green's theorem for $\int_C \mathbf{F} \cdot d\mathbf{r}$ where C is a simple, positively oriented, closed curve in the (x, y) plane and $\mathbf{F}(x, y) = P(x, y)\mathbf{i} + Q(x, y)\mathbf{j}$ is a two dimensional vector field.

(b) Compute the work done by the force field $\mathbf{F}(x, y) = \mathbf{i} + x\mathbf{j}$ on a particle that makes one counterclockwise revolution around the circle $x^2 + y^2 = 1$.

(c) Compute the work done by the force field $\mathbf{F}(x, y) = \mathbf{i} + x\mathbf{j}$ on a particle that travels from $(1, 0)$ to $(0, 1)$, counterclockwise along part of the circle $x^2 + y^2 = 1$.

Solution:

(a) Green's theorem says that

$$\int_C \mathbf{F} \cdot d\mathbf{r} = \int\int_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dA$$

where D is the region enclosed by C.

(b) Directly apply Green's theorem.

$$W = \int_C \mathbf{F} \cdot d\mathbf{r} = \int\int_D (1 - 0)\ dA$$

where D is the circular region $x^2 + y^2 < 1$. The double integral is just the area of D so the answer is $W = \pi$.

(c) The curve is not closed in this case. Therefore Green's theorem does not apply and we have to parameterize the curve. We use polar coordinates for simplicity: $\vec{r}(\theta) = \cos\theta \vec{i} + \sin\theta \vec{j}$ on $0 \le \theta \le \pi/2$.

$$\begin{aligned} W &= \int_C \mathbf{F} \cdot d\mathbf{r} = \int_0^\pi \mathbf{F}(\vec{r}(\theta)) \cdot \vec{r}'(\theta) d\theta \\ &= \int_0^\pi \begin{pmatrix} 1 \\ \cos\theta \end{pmatrix} \cdot \begin{pmatrix} -\sin\theta \\ \cos\theta \end{pmatrix} d\theta = \int_0^\pi \cos^2\theta d\theta \\ &= \int_0^\pi \frac{1}{2} (1 + \cos 2\theta)\ d\theta = \frac{\pi}{2}. \end{aligned}$$