Math 263, Fall 2008
Midterm I Solutions

1. A rectangular box lies in the region $x \geq 0, y \geq 0, z \geq 0$. One corner of the box is at $(0,0,0)$ and another corner lies on the sphere $x^2 + y^2 + z^2 = 3$. Find the maximum possible volume of the box.

(15 points)

Solution. We need to maximize the volume $V = xyz$ subject to the constraint $x^2 + y^2 + z^2 = 3$. The auxiliary function is

$$F(x, y, z; \lambda) = xyz - \lambda(x^2 + y^2 + z^2 - 3),$$

where λ is the Lagrange multiplier. The critical points of F are given by the equations

$$\frac{\partial F}{\partial x} = yz - 2\lambda x = 0,$$
$$\frac{\partial F}{\partial x} = zx - 2\lambda y = 0,$$
$$\frac{\partial F}{\partial x} = xy - 2\lambda z = 0, \text{ and } \frac{\partial F}{\partial \lambda} = x^2 + y^2 + z^2 - 3 = 0.$$

Multiplying the first three equations by x, y, z respectively and adding, we obtain $3xyz = 6\lambda$ or $xyz = 2\lambda$. In view of the first equation, this gives $x^2 = 1$ or $\lambda = 0$. Similarly, we obtain from the second and third equations that either $y^2 = z^2 = 1$ or $\lambda = 0$. But $\lambda = 0$ would imply that $V = xyz = 2\lambda = 0$, which clearly cannot be a maximizer. Therefore $x = y = z = 1$, and the maximum possible volume of the box is 1. □

2. Find all the points on the surface $x^2 + 4y^2 - z^2 = 4$ at which the normal direction to the surface is perpendicular to the plane $2x + 2y + z = 5$.

(15 points)

Proof. The normal vector to the surface at the point (x, y, z) is parallel to $\langle 2x, 8y, -2z \rangle$. We are looking for points that satisfy

$$2x = 2\lambda, \quad 8y = 2\lambda, \quad -2z = \lambda, \quad x^2 + 4y^2 - z^2 = 4.$$

In other words, $\lambda^2 + 4(\lambda/4)^2 - (-\lambda/2)^2 = 4$, or $\lambda = \pm 2$. The points are $(2, \frac{1}{2}, -1)$ and $(-2, \frac{1}{2}, 1)$. □

3. The water temperature in a swimming pool is given by the equation

$$T = x^2 e^y - xy^3.$$

Suppose you swim through the point $(1, 0)$. What is the maximum possible rate of increase in water temperature that you could experience as you swim through that point? In which direction should you swim in order to feel this maximum rate of increase in temperature?

(7 + 8 = 15 points)

Solution. $\nabla T(x, y) = \langle 2xe^y - y^3, x^2 e^y - 3xy^2 \rangle$. The maximum rate of increase in water temperature at $(1, 0)$ is $|\nabla T(1, 0)| = |\langle 2, 1 \rangle| = \sqrt{5}$. The direction in which this maximum increase occurs is $\nabla T / |\nabla T| = \langle 2, 1 \rangle / \sqrt{5}$. □

4. Sketch the domain of integration and evaluate

$$\int_0^4 \int_{\sqrt{x}}^2 \frac{1}{y^3 + 1} \, dy\, dx.$$

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Solution. Interchanging the order of integration, we get

$$
\int_0^4 \int_{\sqrt{x}}^2 \frac{1}{y^3+1}\,dy\,dx = \int_{y=0}^2 \int_{x=0}^{y^2} \frac{1}{y^3+1}\,dx\,dy
$$
$$
= \int_0^2 \frac{y^2}{y^3+1}\,dy
$$
$$
= \frac{1}{3}\ln(y^3+1)\Big|_0^2 = \frac{\ln 9}{3}.
$$

$\square$

5. (a) Can there be a continuously differentiable function f satisfying
$$
f_x = 2xy^2 \quad \text{and} \quad f_y = 2x^2 y?
$$
If so, find such a function. If not, explain clearly why.

(10 points)

Solution. Yes, $f(x,y) = x^2 y^2$ is such a function. $\square$

(b) Can there be a continuously differentiable function f satisfying
$$
f_x = 2x^2 y \quad \text{and} \quad f_y = 2xy^2?
$$
If so, find such a function. If not, explain clearly why.

(10 points)

Solution. There is no such function by Clairaut's theorem, since $f_{xy} = 2x^2 \neq f_{yx} = 2y^2$. $\square$

6. Let C be the curve of intersection of the surface $z = 2x^2 - y^2$ and the plane $z = 4$. Find parametric equations for the tangent line to C at the point $(-2, 2, 4)$.

(15 points)

Solution. Let $(x, y(x), z(x))$ be a parametrization of the curve. Then $z(x)^2 = 2x^2 - y(x)^2$ and $z(x) = 4$. Differentiating both equations implicitly in x, we obtain
$$
2z(x)z'(x) = 4x - 2y(x)y'(x), \quad \text{and} \quad z'(x) = 0.
$$
Setting $x = -2$, $y = 2$ gives $y'(-2) = -8/4 = -2$. The direction of the tangent line is therefore $\langle 1, y'(x), z'(x) \rangle = \langle 1, -2, 0 \rangle$, so that the parametric equation of the line is
$$
x = -2 + t, \quad y = 2 - 2t, \quad z = 4.
$$

$\square$