Math 263 Assignment 7
SOLUTIONS

Problems to turn in:

(1) In each case sketch the region and then compute the volume of the solid region.

(a) The "ice-cream cone" region which is bounded above by the hemisphere $z = \sqrt{a^2 - x^2 - y^2}$ and below by the cone $z = \sqrt{x^2 + y^2}$.

Solution. In spherical coordinates,

$$V = \int_0^{2\pi} \int_0^{\pi/4} \int_0^a \rho^2 \sin\phi d\rho d\phi d\theta = 2\pi \int_0^{\pi/4} \sin\phi \rho^3/3\Big]_0^a d\phi$$
$$= 2\pi\ (-\cos\phi)]_0^{\pi/4}\ a^3/3 = \frac{2\pi a^3}{3}(-\cos\pi/4 + \cos 0) = \frac{\pi a^3(2-\sqrt{2})}{3}$$

or in cylindrical coordinates,

$$V = \int_0^{2\pi} \int_0^{a/\sqrt{2}} \int_r^{\sqrt{a^2-r^2}} r dz dr d\theta = 2\pi \int_0^{a/\sqrt{2}} (r\sqrt{a^2-r^2} - r^2) dr$$
$$= 2\pi \left[\frac{-(a^2-r^2)^{3/2} - r^3}{3}\right]_0^{a/\sqrt{2}} = 2\pi \frac{-(a^2-a^2/2)^{3/2} + (a^2-0^2)^{3/2} - (a/\sqrt{2})^3 + (0)^3}{3}$$
$$= 2\pi \left(\frac{a^3 - 2a^3/2\sqrt{2}}{3}\right) = \frac{\pi a^3(2-\sqrt{2})}{3}.$$

(b) The region bounded by $z = x^2 + 3y^2$ and $z = 4 - y^2$.

Solution. The parabolic cylinder $z = 4 - y^2$ comprises the top of the surface (considered in terms of z) and the paraboloid $z = x^2 + 3y^2$ is the bottom surface in terms of z. To determine the region of the xy-plane which the region bounded by these two surfaces lies over, we intersect the two surfaces, in this case we can set them equal to each other. We see that $x^2 + 3y^2 = 4 - y^2$ if and only if $x^2 + 4y^2 = 4$ if and only if $(x/2)^2 + y^2 = 1$. We will set up and compute the integral of this volume in rectangular coordinates (using a table of integrals to compute the anti-derivative $\int (4-x^2)^{3/2} dx$).

$$V = \int_{-2}^{2} \int_{-\sqrt{4-x^2}/2}^{\sqrt{4-x^2}/2} \int_{x^2+3y^2}^{4-y^2} dz dy dx = \int_{-2}^{2} \int_{-\sqrt{4-x^2}/2}^{\sqrt{4-x^2}/2} (4 - x^2 - 4y^2) dy dx$$
$$= \int_{-2}^{2} \left[(4-x^2)y - (4/3)y^3\right]_{-\sqrt{4-x^2}/2}^{\sqrt{4-x^2}/2} dx = 2\int_{-2}^{2} \left(\frac{(4-x^2)^{3/2}}{2} - \frac{(4-x^2)^{3/2}}{6}\right) dx$$
$$= \frac{2}{3}\int_{-2}^{2} (4-x^2)^{3/2} dx = \frac{2}{3}\left[\frac{x}{8}\left(5\cdot 2^2 - 2x^2\right)\sqrt{4-x^2} + \frac{3\cdot 2^4}{8}\sin^{-1}(x/2)\right)\Bigg]_{-2}^{2}$$
$$= 4(\sin^{-1}(1) - \sin^{-1}(-1)) = 4\pi$$

Another way to compute this integral would be to make a substitution $x = 2u$, so $dx = 2du$ and we would be integrate over a circle of radius 1 in (u, y), which we will call $\widetilde{R}$ whereas the ellipse will be called R. This makes everything much simpler. Lets see what happens.

$$V = \int\int_R \left(\int_{x^2+3y^2}^{4-y^2} dz \right) dA = \int\int_R (4 - x^2 - 4y^2) dx dy = \int\int_{\widetilde{R}} (4 - 4u^2 - 4y^2) 2du dy$$
$$= \int_0^{2\pi} \int_0^1 (4 - 4r^2) 2r dr d\theta = \int_0^{2\pi} d\theta \int_0^1 (8r - 8r^3) dr = 2\pi \left[4r^2 - 2r^4 \right]_0^1 = 2\pi(4-2) = 4\pi$$

(c) A sphere with a cylindrical hole bored through its centre. Specifically, the region inside the sphere $x^2 + y^2 + z^2 = 9$ and outside the cylinder $x^2 + y^2 = 4$.

Solution. A sphere of radius 3 has volume $V_S = 36\pi$. Let V_C denote the volume inside the given sphere and the given cylinder simultaneously. The the volume we want, $V = V_S - V_C$. Let's compute V_C using cylindrical coordinates.

$$V_C = \int_0^{2\pi} \int_0^2 \int_{-(9-r^2)^{1/2}}^{(9-r^2)^{1/2}} r dz dr d\theta = \int_0^{2\pi} \int_0^2 (9 - r^2)^{1/2} 2r dr d\theta$$
$$= 2\pi \left[\frac{-2}{3} (9 - r^2)^{3/2} \right]_0^2 = \frac{4\pi}{3} (9^{3/2} - 5^{3/2}) = 36\pi - \frac{4\pi 5^{3/2}}{3};$$

hence $V = V_S - V_C = \frac{4\pi 5^{3/2}}{3}$.

(2) Switch these integrals to spherical coordinates and compute:

Solution. I_1 is an integral over the top half of a solid sphere of radius 3, centred at the origin.

$$I_1 = \int_{-3}^{3} \int_{-\sqrt{9-x^2}}^{\sqrt{9-x^2}} \int_0^{\sqrt{9-x^2-y^2}} z\sqrt{x^2 + y^2 + z^2}\, dz dy dx$$
$$= \int_{\phi=0}^{\pi/2} \int_{\theta=0}^{2\pi} \int_{\rho=0}^{3} [\rho \cos\phi] \sqrt{\rho^2} \rho^2 \sin\phi d\rho d\theta d\phi$$
$$= \left(\int_0^{\pi/2} \sin\phi \cos\phi d\phi \right) \left(\int_0^{2\pi} d\theta \right) \left(\int_0^3 \rho^4 d\rho \right)$$
$$= \left[\frac{1}{2} \sin^2\phi \right]_0^{\pi/2} (2\pi) \left(\frac{3^5}{5} \right) = \frac{243\pi}{5}$$

I_2 is a solid region contained within $x > 0, y > 0, z > 0$. The solid is above the cone $z = \sqrt{x^2 + y^2}$ and below the sphere $x^2 + y^2 + z^2 = 18$. To check this, note that the cone meets the sphere at the height where $z^2 + z^2 = 18$, $z = 3$, and the ring where they intersect is $x^2 + y^2 = 9$. The angle of the point of the bottom of the cone is $\phi = \pi/4$. Putting this together, we have

$$\begin{aligned}
I_2 &= \int_0^3 \int_0^{\sqrt{9-y^2}} \int_{\sqrt{x^2+y^2}}^{\sqrt{18-x^2-y^2}} (x^2+y^2+z^2)\, dz dy dx \\
&= \int_{\phi=0}^{\pi/4} \int_{\theta=0}^{\pi/2} \int_{\rho=0}^{\sqrt{18}} [\rho^2]\rho^2 \sin\phi d\rho d\theta d\phi \\
&= \left(\int_{\phi=0}^{\pi/4} \sin\phi d\phi\right)\left(\int_{\theta=0}^{\pi/2} d\theta\right)\left(\int_{\rho=0}^{\sqrt{18}} \rho^4 d\rho\right) \\
&= [-\cos\phi]_0^{\pi/4}\left(\frac{\pi}{2}\right)\left(\frac{(3\sqrt{2})^5}{5}\right) = \frac{486\pi}{5}(\sqrt{2}-1)
\end{aligned}$$

(3) Calculate the moment of inertia of a circular pipe of outer radius a, inner radius b, length L and uniform density R, rotating about its centre axis. From your answer, let $b \to 0$ and derive the formula for a solid cylinder too.

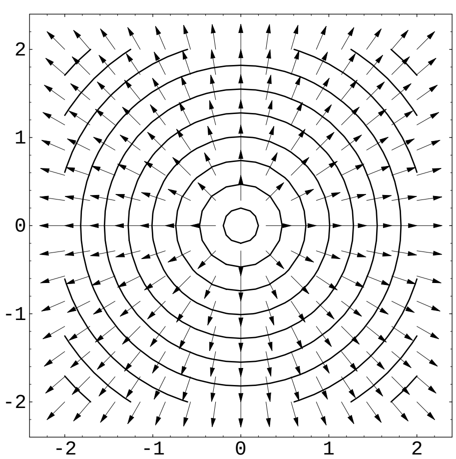
Solution. Line the cylinder up along the z-direction and then the integral we need is easy to do in cylindrical coordinates:

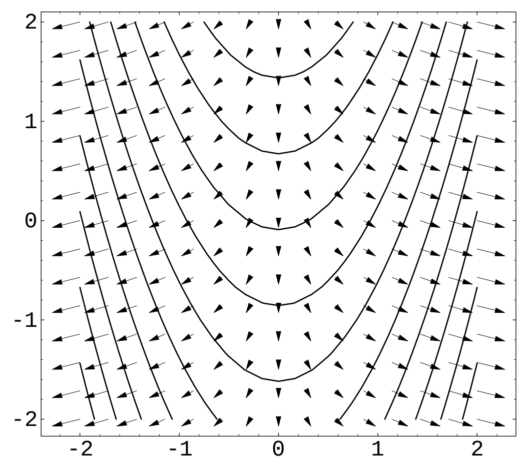
$$\int\int\int (x^2+y^2) R dV = R \int_0^{2\pi} \int_0^L \int_b^a r^2 r dr dz d\theta = \frac{2}{5}\pi LR(a^4 - b^4).$$

Letting $b \to 0$, we obtain the moment of inertia of a solid cylinder, $(2/5)\pi LRa^4$.

(4) Find the gradient vector field of $f(x,y) = \sqrt{x^2+y^2}$ and $g(x,y) = x^2 - y$. In each case, plot the gradient vector field and the contour plot of the function, on the same diagram.

$$\nabla f = \left(\frac{x}{\sqrt{x^2+y^2}}, \frac{y}{\sqrt{x^2+y^2}}\right), \qquad \nabla g = (2x, -1)$$





(5) Compute $\int_C f(x,y,z) ds$ for the following curves and functions.

(a) $C_1 : \mathbf{r}(t) = \langle 30\cos^3 t, 30\sin^3 t\rangle$ for $0 \le t \le \pi/2$ and $f(x,y) = 1 + y/3$.

Solution. First, $ds = |\mathbf{r}'(t)|dt = \sqrt{(-90\cos^2 t \sin t)^2 + (90 \sin^2 t \cos t)^2} dt = 90 \cos t \sin t dt$.
Now we are in a position to compute the line integral.

$$\int_C (1+y/3)ds = \int_0^{\pi/2} (1+10\sin^3 t)90\cos t \sin t dt = \int_0^{\pi/2} (90\sin t + 900\sin^4 t)\cos t dt$$
$$= \int_{u=0}^1 (90u + 900u^4)du, \text{ where } u = \sin t$$
$$= [45u^2 + 180u^5]_0^1 = 225$$

(b) $C_2 : \mathbf{r}(t) = \langle t^2/2, t^3/3\rangle$ for $0 \le t \le 1$ and $f(x,y) = x^2 + y^2$.
Solution. Again we start by computing $ds = |\mathbf{r}'(t)|dt = t\sqrt{1+t^2}dt$. Then

$$\int_C (x^2+y^2)ds = \int_0^1 ((t^2/2)^2 + (t^3/3)^2)t\sqrt{1+t^2}dt = \frac{1}{4}\int_0^1 t^4\sqrt{1+t^2}(tdt) + \frac{1}{9}\int_0^1 t^6\sqrt{1+t^2}(tdt)$$
$$= \frac{1}{8}\int_{u=1}^2 (u-1)^2\sqrt{u}du + \frac{1}{18}\int_{u=1}^2 (u-1)^3\sqrt{u}du, \text{ where } u = 1+t^2$$
$$= \frac{1}{8}\int_1^2 (u^{5/2} - 2u^{3/2} + u^{1/2})du + \frac{1}{18}\int_1^2 (u^{7/2} - 3u^{5/2} + 3u^{3/2} - u^{1/2})du$$
$$= \left[\frac{u^{7/2}}{28} - \frac{u^{5/2}}{10} + \frac{u^{3/2}}{12} + \frac{u^{9/2}}{81} - \frac{u^{7/2}}{21} + \frac{u^{5/2}}{15} - \frac{u^{3/2}}{27}\right]_1^2$$
$$= \left[\frac{u^{9/2}}{81} - \frac{u^{7/2}}{84} - \frac{u^{5/2}}{30} + \frac{5u^{3/2}}{108}\right]_1^2$$
$$= (2^{9/2}/81 - 2^{7/2}/84 - 2^{5/2}/30 + 5\cdot 2^{3/2}/108) - (1/81 - 1/84 - 1/30 + 5/108)$$

(c) $C_3 : \mathbf{r}(t) = \langle 1, 2, t^2\rangle$ for $0 \le t \le 1$ and $f(x,y,z) = e^{\sqrt{z}}$.
Solution.

$$\int_C e^{\sqrt{z}}ds = \int_0^1 e^t\sqrt{0^2+0^2+(2t)^2}dt = \int_0^1 2te^t dt = [2te^t - 2e^t]_0^1 = 2$$

Note that we had to integrate by parts to anti-differentiate $2te^t$. (You let $u = 2t$ and $dv = e^t$.)

(6) Determine whether or not the following vector fields are conservative. In the cases where $\mathbf{F}$ is conservative, find a function φ such that $\mathbf{F}(x,y,z) = \nabla\varphi(x,y,z)$.
(a) $\mathbf{F} = (2xy+z^2)\mathbf{i} + (x^2+2yz)\mathbf{j} + (y^2+2xz)\mathbf{k}$.
Solution. We first test to determine whether or not $\mathbf{F}$ might be conservative. Letting $F_1 = 2xy + z^2$, $F_2 = x^2 + 2yz$, and $F_3 = y^2 + 2xy$ (as usual), it is easy to verify that $\partial F_1/\partial y = \partial F_2/\partial x$, $\partial F_1/\partial z = \partial F_3/\partial x$, and $\partial F_2/\partial z = \partial F_3/\partial y$. There are many ways to find a function $\varphi(x,y,z)$ such that $\nabla\varphi = \mathbf{F}$, which is what we need to find. Here is one method. We will take antiderivatives of F_1 with respect to x, F_2 with respect to y, and F_3 with respect to z respectively and then compare the results.

$$\varphi(x,y,z) = \int (2xy+z^2)dx = x^2y + xz^2 + C_1(y,z)$$
$$\varphi(x,y,z) = \int (x^2+2yz)dy = x^2y + y^2z + C_2(x,z)$$
$$\varphi(x,y,z) = \int (y^2+2xz)dz = y^2z + xz^2 + C_3(x,y)$$

It is very important that $C_1(y,z)$ is function of y and z and not just a constant, since we are "undoing" a partial derivative where we considered y and z as constants (similarly for $C_2(x,z)$ and $C_3(x,y)$). If we examine the three versions of $\varphi(x,y,z)$ we see that each version has at least one term in common. Therefore, we might try $\varphi(x,y,z) = x^2y + y^2z + xz^2$, which turns out to work in this case.

(b) $\mathbf{F} = (\ln(xy))\mathbf{i} + (\frac{x}{y})\mathbf{j} + (y)\mathbf{k}$.

Solution. Note that $\mathbf{F}$ is only defined for $x, y > 0$ or $x, y < 0$ and $F_1 = \ln(xy)$, $F_2 = x/y$, and $F_3 = y$ have continuous partials in these regions of the plane. Further, if $\mathbf{F} = \nabla\varphi$, and hence $\mathbf{F}$ is conservative, then the mixed second partials of φ must be equal. But since $\partial F_2/\partial z = 0$ and $\partial F_3/\partial y = 1$, no such φ could exist with $\nabla\varphi = (\ln(xy))\mathbf{i} + (\frac{x}{y})\mathbf{j} + (y)\mathbf{k}$.

c) $\mathbf{F} = (e^x \cos y)\mathbf{i} + (-e^x \sin y)\mathbf{j} + (2z)\mathbf{k}$.

Solution. By inspection, it is easy to see that $\varphi(x,y,z) = z^2 + e^x \cos y$ is a potential function for $\mathbf{F}$. Otherwise, one could use a method similar to (a).