MATH 263 ASSIGNMENT 1 SOLUTIONS

1) Find the equation of a sphere if one of its diameters has end points $(2,1,4)$ and $(4,3,10)$.

Solution. The centre of the sphere is the midpoint of the diameter, which is $\frac{1}{2}\big[(2,1,4)+(4,3,10)\big] = (3,2,7)$. The length of the diameter is $\sqrt{\big|(4,3,10)-(2,1,4)\big|^2} = \sqrt{2^2+2^2+6^2} = \sqrt{44}$ so the radius of the sphere is $\frac{1}{2}\sqrt{44} = \sqrt{11}$. The equation of the sphere is $\boxed{(x-3)^2+(y-2)^2+(z-7)^2 = 11}$

2) Show that the set of all points P that are twice as far from $(-1,5,3)$ as from $(6,2,-2)$ is a sphere. Find its centre and radius.

Solution. Let the coordinates of a point P be (x,y,z). This point is twice as far from $(-1,5,3)$ as from $(6,2,-2)$ if and only if

$$\begin{aligned}
&\sqrt{(x+1)^2+(y-5)^2+(z-3)^2} = 2\sqrt{(x-6)^2+(y-2)^2+(z+2)^2}\\
\iff& (x+1)^2+(y-5)^2+(z-3)^2 = 4(x-6)^2+4(y-2)^2+4(z+2)^2\\
\iff& x^2+2x+1+y^2-10y+25+z^2-6z+9 = 4x^2-48x+144+4y^2-16y+16+4z^2+16z+16\\
\iff& 3x^2-50x+3y^2-6y+3z^2+22z+141 = 0\\
\iff& 3\big(x-\tfrac{25}{3}\big)^2+3\big(y-1\big)^2+3\big(z+\tfrac{11}{3}\big)^2+141-\tfrac{625}{3}-3-\tfrac{121}{3} = 0\\
\iff& \big(x-\tfrac{25}{3}\big)^2+\big(y-1\big)^2+\big(z+\tfrac{11}{3}\big)^2 = \tfrac{332}{9}
\end{aligned}$$

This is a circle of $\boxed{\text{centre } \big(\tfrac{25}{3},1,-\tfrac{11}{3}\big)}$ and $\boxed{\text{radius } \tfrac{\sqrt{332}}{3}}$.

3) Describe and sketch the set of all points in $\mathbb{R}^3$ that satisfy

a) $x^2+y^2+z^2 = 2z$ b) $x^2+z^2 = 4$ c) $z \ge \sqrt{x^2+y^2}$

d) $x^2+y^2+z^2 = 4,\ z = 1$ e) $x+y+z = 1$

Solution.

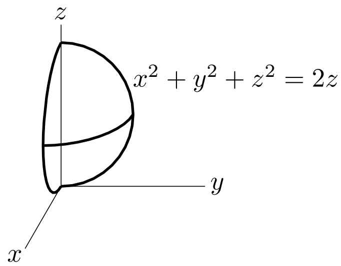
a) Since $x^2+y^2+z^2 = 2z$ is equivalent to $x^2+y^2+(z-1)^2 = 1$, this is the set of points whose distance from $(0,0,1)$ is 1. So this is the sphere of radius 1 centred on $(0,0,1)$.

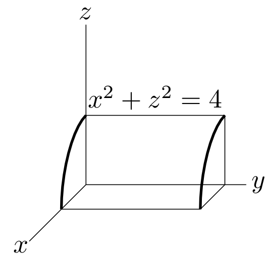
b) For each fixed $y_0 \ge 0$, the curve $x^2+z^2 = 4,\ y = y_0$ is a circle in the plane $y = y_0$ with centre $(0,y_0,0)$ and radius 2. As $x^2+z^2 = 4$ is the union of $x^2+z^2 = 4,\ y = y_0$ for all possible values of y_0, it is a horizontal stack of vertical circles. The surface is the cylinder of radius 2 centred on the y–axis.

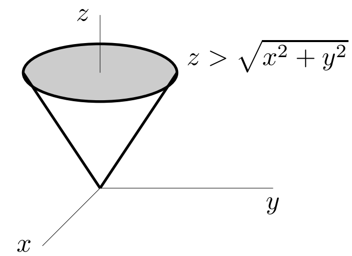
c) For each fixed $z_0 \ge 0$, the curve $z = \sqrt{x^2+y^2},\ z = z_0$ is a circle in the plane $z = z_0$ with centre $(0,0,z_0)$ and radius z_0. As $\sqrt{x^2+y^2} = z$ is the union of $\sqrt{x^2+y^2} = z,\ z = z_0$ for all possible values of $z_0 \ge 0$, it is a vertical stack of horizontal circles whose radii increase linearly with z. It is a cone centered on the z–axis. $z > \sqrt{x^2+y^2}$ is the region above this cone. It is a solid cone.

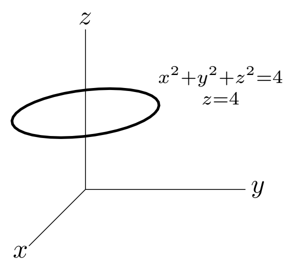
d) This is the circle of radius $\sqrt{3}$ centred on $(0,0,1)$ that lies parallel to the xy–plane.

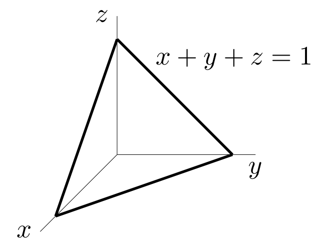
e) This is the plane which passes through the points $(1,0,0)$, $(0,1,0)$ and $(0,0,1)$.











4) Compute the dot product of the vectors $\vec{a}$ and $\vec{b}$. Find the angle between them.

a) $\vec{a} = \langle -1, 1\rangle$, $\vec{b} = \langle 1, 1\rangle$ b) $\vec{a} = \langle 1, 1\rangle$, $\vec{b} = \langle 2, 2\rangle$

Solution.

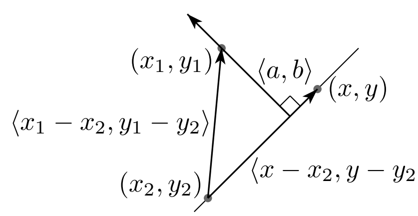
$$\text{a)}\quad \vec{a}\cdot\vec{b} = \langle -1,1\rangle\cdot\langle 1,1\rangle = 0 \quad \cos\theta = \frac{0}{\sqrt{2}\sqrt{2}} = 0 \quad \theta = 90^\circ$$

$$\text{b)}\quad \vec{a}\cdot\vec{b} = \langle 1,1\rangle\cdot\langle 2,2\rangle = 4 \quad \cos\theta = \frac{4}{\sqrt{2}\sqrt{8}} = 1 \quad \theta = 0^\circ$$

5) Use a projection to derive a formula for the distance from a point (x_1, y_1) to the line $ax + by = c$. Here, a and b are not both zero.

Solution. Let (x_2, y_2) be any point on the line. Then $ax_2 + by_2 = c$. If (x, y) is any other point on the line, then $ax + by = c$ so that $a(x_2 - x) + b(y_2 - y) = c - c = 0$. That is, $\langle a, b\rangle$ is perpendicular to $\langle x_2 - x, y_2 - y\rangle$. As $\langle x_2 - x, y_2 - y\rangle$ is an arbitrary vector lying on the line, $\langle a, b\rangle$ is a normal to the line. The distance from (x_1, y_1) to $ax + by = c$ is the length of the projection of the vector $\langle x_1 - x_2, y_1 - y_2\rangle$ on the vector $\langle a, b\rangle$, which is

$$\frac{|\langle x_1-x_2, y_1-y_2\rangle\cdot\langle a,b\rangle|}{|\langle a,b\rangle|} = \frac{|ax_1 - ax_2 + by_1 - by_2|}{\sqrt{a^2+b^2}} = \boxed{\frac{|ax_1+by_1-c|}{\sqrt{a^2+b^2}}}$$



6) Compute $\langle 1, 2, 3\rangle \times \langle 4, 5, 6\rangle$.

Solution.

$$\langle 1,2,3\rangle\times\langle 4,5,6\rangle = \det\begin{bmatrix} \hat{\imath} & \hat{\jmath} & \hat{\mathbf{k}} \\ 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix} = \hat{\imath}(2\times 6 - 3\times 5) - \hat{\jmath}(1\times 6 - 3\times 4) + \hat{\mathbf{k}}(1\times 5 - 2\times 4) = \boxed{-3\hat{\imath} + 6\hat{\jmath} - 3\hat{\mathbf{k}}}$$

7) Prove that

a) $\hat{\imath}\times\hat{\jmath} = \hat{\mathbf{k}}$

b) $\vec{a}\cdot(\vec{a}\times\vec{b}) = \vec{b}\cdot(\vec{a}\times\vec{b}) = 0$

c) $|\vec{a}\times\vec{b}|^2 = |\vec{a}|^2\,|\vec{b}|^2 - (\vec{a}\cdot\vec{b})^2$

Solution. a)

$$\hat{\imath}\times\hat{\jmath} = \det\begin{bmatrix} \hat{\imath} & \hat{\jmath} & \hat{\mathbf{k}} \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} = \hat{\imath}(0\times 0 - 0\times 1) - \hat{\jmath}(1\times 0 - 0\times 0) + \hat{\mathbf{k}}(1\times 1 - 0\times 0) = \boxed{\hat{\mathbf{k}}}$$

b)

$$\vec{a}\cdot(\vec{a}\times\vec{b}) = a_1\big(a_2b_3 - a_3b_2\big) - a_2\big(a_1b_3 - a_3b_1\big) + a_3\big(a_1b_2 - a_2b_1\big) = 0$$

$$\vec{b}\cdot(\vec{a}\times\vec{b}) = b_1\big(a_2b_3 - a_3b_2\big) - b_2\big(a_1b_3 - a_3b_1\big) + b_3\big(a_1b_2 - a_2b_1\big) = 0$$

c) Just compare

$$|\vec{a}\times\vec{b}|^2 = (a_2b_3 - a_3b_2)^2 + (a_3b_1 - a_1b_3)^2 + (a_1b_2 - a_2b_1)^2$$
$$= a_2^2b_3^2 - 2a_2b_3a_3b_2 + a_3^2b_2^2 + a_3^2b_1^2 - 2a_3b_1a_1b_3 + a_1^2b_3^2 + a_1^2b_2^2 - 2a_1b_2a_2b_1 + a_2^2b_1^2$$

and

$$|\vec{a}|^2|\vec{b}|^2 - (\vec{a}\cdot\vec{b})^2 = \big(a_1^2 + a_2^2 + a_3^2\big)\big(b_1^2 + b_2^2 + b_3^2\big) - \big(a_1b_1 + a_2b_2 + a_3b_3\big)^2$$
$$= a_1^2b_2^2 + a_1^2b_3^2 + a_2^2b_1^2 + a_2^2b_3^2 + a_3^2b_1^2 + a_3^2b_2^2 - \big(2a_1b_1a_2b_2 + 2a_1b_1a_3b_3 + 2a_2b_2a_3b_3\big)$$

8) Find the equation of the sphere which has the two planes $x+y+z=3,\ x+y+z=9$ as tangent planes if the centre of the sphere is on the planes $2x-y=0,\ 3x-z=0$.

Solution. The planes $x+y+z=3$ and $x+y+z=9$ are parallel. So the centre lies on $x+y+z=6$ (the plane midway between $x+y+z=3$ and $x+y+z=9$) as well as on $y=2x$ and $z=3x$. Solving,

$$y=2x,\ z=3x,\ x+y+z=6 \ \Rightarrow\ x+2x+3x=6 \ \Rightarrow\ x=1,\ y=2,\ z=3$$

So the centre is at $(1,2,3)$. The normal to $x+y+z=3$ is $\langle 1,1,1\rangle$. The points $(1,1,1)$ on $x+y+z=3$ and $(3,3,3)$ on $x+y+z=9$ differ by a vector, $\langle 2,2,2\rangle$, which is a multiple of this normal. So the distance between the planes is $|\langle 2,2,2\rangle| = 2\sqrt{3}$ and the radius of the sphere is $\sqrt{3}$. The sphere is

$$\boxed{(x-1)^2 + (y-2)^2 + (z-3)^2 = 3}$$

9) Find the equation of the plane that passes through the point $(-2,0,-1)$ and through the line of intersection of $2x+3y-z=0,\ x-4y+2z=-5$.

Solution. First we'll find two points on the line of intersection of $2x+3y-z=0,\ x-4y+2z=-5$. This will give us three points on the plane.

$$\left\{\begin{aligned} 2x+3y-z &= 0 \\ x-4y+2z &= -5 \end{aligned}\right\} \iff \left\{\begin{aligned} 2x+3y &= z \\ x-4y &= -2z-5 \end{aligned}\right\} \iff \left\{\begin{aligned} 2x+3y &= z \\ 11y &= 5(z+2) \end{aligned}\right\}$$

In the last step, we subtracted twice the second equation from the first. So if $z=-2$, then $y=0$ and $x=-1$. And if $z=-\frac{15}{2}$, then $y=-\frac{5}{2}$ and $x=0$. So we conclude that the three points $(-2,0,-1)$, $(-1,0,-2)$ and $(0,-\frac{5}{2},-\frac{15}{2})$ must all lie on the plane. So the two vectors $\langle -2,0,-1\rangle - \langle -1,0,-2\rangle = \langle -1,0,1\rangle$ and $\big\langle 0,-\frac{5}{2},-\frac{15}{2}\big\rangle - \langle -1,0,-2\rangle = \big\langle 1,-\frac{5}{2},-\frac{11}{2}\big\rangle$ must be parallel to the plane. So the normal to the plane is $\langle -1,0,1\rangle \times \big\langle 1,-\frac{5}{2},-\frac{11}{2}\big\rangle = \big\langle \frac{5}{2},-\frac{9}{2},\frac{5}{2}\big\rangle$ or, equivalently $\vec{n} = \langle 5,-9,5\rangle$. The equation of the plane is

$$5(x+2) - 9y + 5(z+1) = 0 \text{ or } \boxed{5x-9y+5z=-15}$$

10) Find the equations of the line through $(2,-1,-1)$ and parallel to each of the two planes $x+y=0$ and $x-y+2z=0$. Express the equations of the line in vector and scalar parametric forms and in symmetric form.

Solution. One vector normal to $x+y=0$ is $\langle 1,1,0\rangle$. One vector normal to $x-y+2z=0$ is $\langle 1,-1,2\rangle$. The vector $\langle 1,-1,-1\rangle$ is perpendicular to both of those normals and hence is parallel to both planes. So $\langle 1,-1,-1\rangle$ is also parallel to the line. The vector parametric equation of the line is

$$\boxed{\vec{x} = (2,-1,-1) + t(1,-1,-1)}$$

The scalar parametric equations of the line are

$$\boxed{x=2+t,\ y=-1-t,\ z=-1-t}$$

The symmetric equations are

$$t = \boxed{x-2 = -y-1 = -z-1}$$