L^p DECAY ESTIMATES FOR WEIGHTED OSCILLATORY INTEGRAL OPERATOR ON $\mathbb{R}$

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Abstract. In this paper, we formulate necessary conditions for decay rates of L^p operator norms of weighted oscillatory integral operators on $\mathbb{R}$ and give sharp L^2 estimates and nearly sharp L^p estimates.

1. Introduction

Suppose f and g are real-analytic, real-valued funtions in a neighborhood V of the origin in $\mathbb{R}^2$ with $f(0,0) = g(0,0) = 0$ and let χ be a smooth function of compact support in V. We consider the oscillatory integral operator

$$T_\lambda \varphi(x) = \int_{\mathbb{R}} e^{i\lambda f(x,y)} |g(x,y)|^{\epsilon/2} \chi(x,y) \varphi(y) dy,$$

where ϵ is any positive number. In this paper we will study the decay rate in λ of $||T_\lambda||_{L^p \to L^p}$ as $\lambda \to \infty$.

The case where $g(x,y) = 1$ has been studied in [Gr], [GS3], [PSt1], [PSt2], [PSt3], and [R]. In [PSt1] and [PSt2], Phong and Stein considered a case where the phase function $f(x,y)$ is a real homogeneous polynomial and they obtained sharp decay estimates for $||T_\lambda||_{L^2 \to L^2}$. In [PSt3], they took into account of the more general case where the phase function $f(x,y)$ is a real analytic function and they proved $||T_\lambda||_{L^2 \to L^2} \sim \lambda^{-\delta}$ where δ is the reduced Newton distance of $f(x,y)$. In

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[R] Rychkov developed the ideas of Phong and Stein in [PSt3] and Seeger in [S1] to obtain sharp L^2 decay estimates for the case where the phase function $f(x,y)$ is a real smooth function with the condition that the formal power series expansion of f''_{xy} at the origin does not vanish. He proved $||T_\lambda||_{L^2\to L^2} \sim \lambda^{-\delta}$, where δ is the reduced Newton distance of the formal power series expansion of $f(x,y)$ at the origin, with a loss of a certain power of $\log \lambda$ in the case where all solutions $r(x)$ of $f''_{xy}(x, r(x)) = 0$ have the same asymtotic fractional power series expansion with the leading power 1. In [Gr], Greenblatt gave a new proof for the theorem of Phong and Stein in [PSt3]. For L^p estimates, Greenleaf and Seeger obtained sharp decay estimates [GS3]. They considered oscillatory integral operators in $\mathbb{R}^n$ with a real smooth phase function with the assumption of two-sided fold singularities. They established sharp $L^p - L^q$ decay estimates of the oscillatory integral operators. In [S2], Seeger formulated optimal L^p regularity of generalized Radon transforms on $\mathbb{R}^2$ and he obtained sharp L^p regularity estimates except endpoints. In [Y], sharp L^p decay estimates for T_λ have been established excluding two end-point estimates.

The case where $g(x,y) = f''_{xy}(x,y)$ has been studied in [PSt4]. In [PSt4] Phong and Stein proved best possible decay estimate, that is, $||T_\lambda||_{L^2\to L^2} \sim \lambda^{-1/2}$ when $g(x,y) = f''_{xy}(x,y)$ and $\epsilon = 1/2$. We wish to investigate the improvement in the decay rate of $||T_\lambda||_{L^p\to L^p}$ when f is unrelated to g.

Higher dimensional case even without any damping factor has not been understood well. There have been a few L^2 estimates of special cases [C1], [C2], [GS3], [GS4], [PaSo]. Sharp L^2 estimates under the assumption of two-sided fold singularities were obtained in [PaSo]. Optimal estimates with one-sided fold singularity have been established in [C2] and [GS1]. Various types of higher order singularities have been treated in [C1], [GS2], and [GS4]. We recommend [GS5] as a more detailed and organized survey on this subject.

The case where the weight $g(x,y)$ is not related to $f(x,y)$ has been considered by the first author in a different context [Pr]. In [Pr] she introduced weighted Newton distance to treat the weighted integral. We shall use some notions in [Pr] and we briefly describe them. We start with factorizing $f''_{xy}(x,y)$ and $g(x,y)$

$$\begin{aligned} f''_{xy}(x,y) &= U_1(x,y)x^{\alpha_1}y^{\beta_1}\prod(y-r_\nu(x)) \\ g(x,y) &= U_2(x,y)x^{\alpha_2}y^{\beta_2}\prod(y-s_\mu(x)) \end{aligned}$$

where U_i $i=1,2$ are real analytic functions with $U_i(0,0)\neq 0$, α_i's and β_i's are non-negative integers and $r_\nu(x)$'s and $s_\mu(x)$'s are Puiseux series of the form

$$\begin{aligned} r_\nu(x) &= c_\nu x^{a_\nu} + O(x^{b_\nu}) \\ s_\mu(x) &= c_\mu x^{a_\mu} + O(x^{b_\nu}) \end{aligned}$$

where $b_\eta > a_\eta$ are rational numbers and $c_\eta \neq 0$. We reindex the combined set of distinct exponents a_ν and a_μ into increasing order so that

$$0 < a_1 < a_2 < \cdots < a_N.$$

We define

$$m_l = \#\{\nu \;:\; r_\nu(x) = c_\nu x^{a_l} + \cdots, \quad c_\nu \neq 0\}$$

$$n_l = \#\{\mu \;:\; r_\mu(x) = c_\mu x^{a_l} + \cdots, \quad c_\mu \neq 0\}$$

and we call m_l and n_l the generalized multiplicity of f''_{xy} and g, respectively, corresponding to the exponent a_l. Now we define

$$\begin{aligned} A_l &= \alpha_1 + \textstyle\sum_{i=1}^{l} a_l m_l, & B_l &= \beta_1 + \textstyle\sum_{i=l+1}^{N} m_i \\ C_l &= \alpha_2 + \textstyle\sum_{i=1}^{l} a_l n_l, & D_l &= \beta_2 + \textstyle\sum_{i=l+1}^{N} n_i. \end{aligned}$$

Then $\{(A_l, B_l)\}$ and $\{(C_l, D_l)\}$ are sets of vertices of the Newton diagrams of f''_{xy} and g, respectively. The number of common roots of f''_{xy} and g is an important information to obtain optimal estimates. To extract the information we use a coordinate transformation η given by

$$(x, y) \mapsto (x, y - q(x)) \quad \text{or} \quad (x, y) \mapsto (x - q(x), y),$$

where q is a convergent real-valued Puiseux series in a neighborhood of the origin. For $f''_{xy} \circ \eta$ and $g \circ \eta$ we can define previous notions such as A_l, B_l, C_l, D_l, and a_l in the same way. To avoid the confusion we use the notations $A_l(\eta)$, $B_l(\eta)$, $C_l(\eta)$, $D_l(\eta)$, and $a_l(\eta)$ to specify the coordinate transformation η. To describe optimal decay rate of $||T_{\lambda,\epsilon}||_{L^p \to L^p}$ we shall need the following notations. Let $K = [0, 1] \times \mathbb{R}$. We define subsets $\mathcal{A}_0$, $\mathcal{A}_l$, and $\mathcal{A}_{l'}(\eta)$ of K as

$$\begin{aligned} \mathcal{A}_0 &= \left\{ (\frac{1}{p}, \alpha) \in K : \alpha \le \frac{1}{p}, \text{ and } \alpha \le 1 - \frac{1}{p} \right\}, \\ \mathcal{A}_l &= \left\{ (\frac{1}{p}, \alpha) \in K : \alpha \le \frac{\epsilon(C_l + a_l D_l) + 2a_l}{2C} + \frac{1 - a_l}{C} \frac{1}{p} \right\}, \\ \mathcal{A}_{l'}(\eta) &= \left\{ (\frac{1}{p}, \alpha) \in K : \alpha \le \frac{\epsilon(C_{l'}(\eta) + a_{l'}(\eta) D_{l'}(\eta)) + 2a_{l'}(\eta)}{2C'} + \frac{1 - a_l}{C'} \frac{1}{p} \right\} \end{aligned}$$

where $C = 1 + a_l + A_l + a_l B_l$ and $C' = A_{l'}(\eta) + a_{l'}(\eta) B_{l'}(\eta) + 1 + 2a_{l'}(\eta) - a_l$. Set

$$\mathcal{A}_1 = \bigcap_l \mathcal{A}_l \quad \text{and} \quad \mathcal{A}_2 = \bigcap_\eta \bigcap_{l'} \mathcal{A}_{l'}(\eta).$$

Now we finally define $\mathcal{A}$ as

$$\mathcal{A} = \mathcal{A}_0 \cap \mathcal{A}_1 \cap \mathcal{A}_2.$$

Remark 1.1. When we define $A_{l'}(\eta)$ we include the case where $a_{l'}(\eta) = \infty$. In this case we assume that

$$A_{l'}(\eta) = \left\{ (\frac{1}{p}, \alpha) \in K : \alpha \leq \frac{\epsilon D_{l'}(\eta) + 2}{2(B_{l'}(\eta) + 2)} \right\}. \tag{1.1}$$

Theorem 1.2 (Necessity)**.** *If T_λ is bounded on $L^p(\mathbb{R})$ with $||T_\lambda||_{L^p \to L^p} \leq O(\lambda^{-\alpha})$, then $(1/p, \alpha) \in \mathcal{A}$.*

Theorem 1.3 (L^2 estimates)**.** *If $(1/2, \alpha) \in \mathcal{A}$, then $||T_\lambda||_{L^2 \to L^2} \leq O(\lambda^{-\alpha})$*

Theorem 1.4 (L^p estimates)**.** *If $(1/p, \alpha) \in \mathrm{int}(\mathcal{A})$, then we have*

$$||T_\lambda||_{L^p \to L^p} \leq O(\lambda^{-\alpha}).$$

Remark 1.5. In theorem 1.4 we only have estimates in the interior of $\mathcal{A}$. During the proof of the theorem one can easily observe that we have estimates on some part of the boundary of $\mathcal{A}$. We shall discuss this in detail in part 1 of the final remark.

2. Proof of theorem 1.2

Proof of Theorem 1.2. Suppose that T_λ is bounded on L^p with $||T_\lambda||_{L^p \to L^q} \leq O(\lambda^{-\alpha})$. We shall show that $(1/p, \alpha) \in \mathcal{A}$. Suppose $f''_{xy}(x, y) = \sum_{p,q \geq 0} c_{pq} x^p y^q$. Then we have

$$\begin{aligned} f(x, y) &= \sum_{p,q \geq 0} c_{pq} \frac{x^{p+1} y^{q+1}}{(p+1)(q+1)} + F_1(x) + F_2(y) \\ &= \sum_{p,q \geq 1} \tilde{c}_{pq} x^p y^q + F_1(x) + F_2(y) \end{aligned}$$

where $F_1(x)$ and $F_2(y)$ are real analytic. Note that the Newton diagram of $\sum_{p,q \geq 1} \tilde{c}_{pq} x^p y^q$ is same as the reduced Newton diagram of f. We fix l. Let $C =$

$1 + A_l + a_l(B_l + 1)$ and $R > 0$ is a constant to be specified. Now, for large positive λ, we define the function φ_λ, ψ_λ by

$$\varphi_\lambda(y) = \begin{cases} e^{-i\lambda F_2(y)} & \text{if } R \leq y\lambda^{a_l/C} \leq R + c_1 \\ 0 & \text{otherwise,} \end{cases}$$

and

$$\psi_\lambda(x) = \begin{cases} e^{-i\lambda F_1(x)} & \text{if } R \leq x\lambda^{1/C} \leq R + c_1 \\ 0 & \text{otherwise.} \end{cases}$$

We claim that for any $\epsilon > 0$, in the support of $\varphi_\lambda(y)\psi_\lambda(x)$ we have:

$$\left|\lambda f(x,y) - \lambda F_1(x) - \lambda F_2(y) - \sum{}' \tilde{c}_{pq} R^q\right| < \epsilon,$$

where the sum $\sum'$ is taken over (p, q) that belong to the face of the reduced Newton diagram with equation $p + a_l q = C$, as long as c_1 is taken to be small in terms of $\sum' |\tilde{c}_{pq}| R^q$ and then λ is taken to be large. To prove the claim, first we note that if $0 < c_1 < R$ is sufficiently small then we have

$$\begin{aligned} \left|\sum{}' \tilde{c}_{pq}(\lambda x^p y^q - R^q)\right| &\leq \sum{}' |\tilde{c}_{pq}||\lambda x^p y^q - 1| \\ &\leq \sum{}' |\tilde{c}_{pq}|[(1 + \frac{c_1}{R})^q (1 + c_1)^p - 1] \cdot R^q \\ &< \frac{\epsilon}{2}. \end{aligned}$$

Also, because of the convex nature of the Newton diagram, $p + a_l q > C$ for all other (p, q) such that $\tilde{c}_{pq} \neq 0$, so,

$$\lambda \left| \sum_{(p,q); p + a_l q \neq C} \tilde{c}_{pq} x^p y^q \right| < \frac{\epsilon}{2}.$$

If we take, say $\epsilon < \pi/2$ then this shows that

$$\begin{aligned}
| < T_\lambda \varphi_\lambda, \psi_\lambda > | &= \left| \int_{\mathbb{R}^2} e^{i\lambda f(x,y)} |g(x,y)|^{\frac{\epsilon}{2}} \chi(x,y) \varphi_\lambda(y) \overline{\psi_\lambda(x)} dy dx \right| \\
&= \left| \int_{(x,y)\in S_\lambda} e^{i[\lambda f(x,y) - \lambda F_1(x) - \lambda F_2(y)]} \chi(x,y) |g(x,y)|^{\frac{\epsilon}{2}} dy dx \right| \\
&= \left| \int_{(x,y)\in S_\lambda} e^{i[\lambda f(x,y) - \lambda F_1(x) - \lambda F_2(y) - \sum' \tilde{c}_{pq} R^q]} |g(x,y)|^{\frac{\epsilon}{2}} dy dx \right|
\end{aligned}$$

where $S_\lambda = \{(x,y) | 1 \le \lambda^{1/C} x \le 1 + c_1, R \le y \lambda^{a_l/C} \le R + c_1\}$. Hence we have

$$| < T_\lambda \varphi_\lambda, \psi_\lambda > | \ge C \int_{(x,y)\in S_\lambda} \chi(x,y) |g(x,y)|^{\frac{\epsilon}{2}} dy dx.$$

Let $R > 2 \cdot \max\{|c|; y = cx^{a_l} + \cdots$ is a root of $g\}$ and $R > 1$. Then $g(x,y) \sim |x|^{C_l} |y|^{D_l}$ on the support of $\varphi_\lambda(y)\psi_\lambda(x)$. We therefore have

$$| < T_\lambda \varphi_\lambda, \psi_\lambda > | \ge C \lambda^{-\frac{C_l + a_l D_l}{C} \cdot \frac{\epsilon}{2}} \lambda^{-\frac{a_l + 1}{C}}$$

as $\lambda \to \infty$. Hence, we have

$$\begin{aligned}
\frac{| < T_\lambda \varphi_\lambda, \psi_\lambda > |}{||\varphi_\lambda||_p \cdot ||\psi_\lambda||_{p'}} &\ge C \frac{\lambda^{-\frac{C_l + a_l D_l}{C} \cdot \frac{\epsilon}{2}} \lambda^{-\frac{a_l + 1}{C}}}{\lambda^{-\frac{a_l}{Cp} - \frac{1}{C}(1 - \frac{1}{p})}} \\
&\ge C \lambda^{-\frac{\epsilon(C_l + a_l D_l) + 2a_l}{2C} - \frac{1 - a_l}{C} \frac{1}{p}},
\end{aligned}$$

which implies

$$\alpha \le \frac{\epsilon(C_l + a_l D_l) + 2a_l}{2C} + \frac{1 - a_l}{C} \frac{1}{p}.$$

Therefore $(1/p, \alpha) \in \mathcal{A}_1$.

We fix a root $y = r(x)$ in $S_l = \{r_i(x) | r_i(x) = cx^{a_l} + \cdots\}$ and l'. Suppose $r(x) = \tilde{r}(x) + O(|x|^{a_{l'}(\eta)})$. We define φ_λ and ψ_λ as

$$\varphi_\lambda(y) = \begin{cases} e^{-i\lambda F_2(y)} & \text{if } \tilde{r}(\lambda^{-1/C'}) + R\lambda^{-a_{l'}(\eta)/C'} \le y \le \tilde{r}(\lambda^{-1/C'}) + 2R\lambda^{-a_{l'}(\eta)/C'} \\ 0 & \text{otherwise,} \end{cases}$$

and

$$\psi_\lambda(x) = \begin{cases} e^{-i\lambda F_1(x)} & \text{if } \lambda^{-1/C'} \le x \le \lambda^{-1/C'} + c_1\lambda^{-(a_{l'}(\eta)-a_l+1)/C'} \\ 0 & \text{otherwise} \end{cases}$$

where $C' = A_{l'}(\eta) + a_{l'}(\eta)B_{l'}(\eta) + 1 + 2a_{l'}(\eta) - a_l$, c_1 and R are constants, and F_1, F_2 are real-valued functions to be specified later. On the support of $\varphi_\lambda(y)\psi_\lambda(x)$ we have

$$|y - r(x)| \le |\tilde{r}(\lambda^{-1/C'}) + 2R\lambda^{-a_{l'}(\eta)/C'} - \tilde{r}(x) + O(\lambda^{-a_{l'}(\eta)/C'})|.$$

Suppose $\tilde{r}(x) = \alpha x^{a_l} + \beta x^{b_l} + \cdots$ where without loss of generality $\alpha > 0, \beta > 0$. Then

$$\begin{aligned} |y - r(x)| &\le |\alpha\lambda^{-a_l/C'} + \beta\lambda^{-b_l/C'} + 2R\lambda^{-a_{l'}(\eta)/C'} - \alpha\lambda^{-a_l/C'} \\ &\quad - \beta[\lambda^{-1/C'}(1 + c_1\lambda^{-(a_{l'}(\eta)-a_l)/C'} + O(\lambda^{-a_{l'}(\eta)/C'})]| \\ &\le 3R\lambda^{-a_{l'}(\eta)/C'} \end{aligned}$$

and

$$\begin{aligned} |y - r(x)| &\ge |R\lambda^{-a_{l'}(\eta)/C'} + \tilde{r}(\lambda^{-1/C'}) - \alpha[\lambda^{-1/C'}(1 + c_1\lambda^{-(a_{l'}(\eta)-a_l)/C'})]^{a_l} \\ &\quad - \beta\lambda^{-b_l/C'} + o(\lambda^{-a_{l'}(\eta)/C'})| \ge \frac{R}{2}\lambda^{-a_{l'}(\eta)/C'}. \end{aligned}$$

Let $(x_0(\lambda), y_0(\lambda))$ be a fixed point on the support of $\varphi_\lambda(y)\psi_\lambda(x)$. Then for any (x, y) in the support

$$\begin{aligned} \int_{x_0}^{x}\int_{y_0}^{y} f''_{xy}(s,t)\,dt\,ds &= \int_{x_0}^{x} [f'_x(s,y) - f'_x(s,y_0)]\,ds \\ &= f(x,y) - f(x_0,y) - f(x,y_0) + f(x_0,y_0). \end{aligned} \tag{2.1}$$

Let $F_2(y) = f(x_0(\lambda), y)$, $F_1(x) = f(x, y_0(\lambda)) - f(x_0(\lambda), y_0(\lambda))$. We notice that for (s,t) in the support of $\varphi_\lambda(y)\psi_\lambda(x)$,

$$\begin{aligned} f''_{xy}(s,t) &\sim |t - \tilde{r}(s)|^{B_{l'}(\eta)}|s|^{A_{l'}(\eta)} \\ &\sim R^{B_{l'}(\eta)}\lambda^{-\frac{A_{l'}(\eta)+a_{l'}(\eta)B_{l'}(\eta)}{C'}} \end{aligned}$$

Therefore we have

$$\begin{aligned} \int_{x_0}^{x}\int_{y_0}^{y} f''_{xy}(s,t)dtds &\sim R^{B_{l'}(\eta)+1}\lambda^{-\frac{A_{l'}(\eta)+a_{l'}(\eta)B_{l'}(\eta)}{C'}}\lambda^{-\frac{a_{l'}(\eta)}{C'}}\cdot c_1\lambda^{-\frac{a_{l'}(\eta)-a_l+1}{C'}} \\ &\sim R^{B_{l'}(\eta)+1}\cdot c_1\cdot\lambda^{-1} \end{aligned}$$

By choosing c_1 sufficiently small, we can ensure that for some $0 < \epsilon < \pi/4$

$$|\lambda f(x,y) - \lambda f(x_0,y) - \lambda f(x,y_0) + \lambda f(x_0,y_0)| < \epsilon.$$

Hence we have

$$\begin{aligned} |< T_\lambda\varphi_\lambda, \psi_\lambda >| &\geq \int_{(x,y)\in S_\lambda} |g(x,y)|^{\frac{\epsilon}{2}}dydx \\ &\geq \lambda^{-\frac{\epsilon(C_{l'}(\eta)+a_{l'}(\eta)D_{l'}(\eta))}{2C'}}\lambda^{-\frac{a_{l'}(\eta)}{C'}}\lambda^{-\frac{a_{l'}(\eta)-a_l+1}{C'}}. \end{aligned}$$

This yields

$$\frac{|< T_\lambda\varphi_\lambda, \psi_\lambda >|}{||\varphi_\lambda||||\psi_\lambda||} \geq C\lambda^{-\frac{\epsilon(C_{l'}(\eta)+a_{l'}(\eta)D_{l'}(\eta))+2a_{l'}(\eta)}{2C'}-\frac{1-a_l}{C'}\frac{1}{p}},$$

which implies

$$\alpha \leq \frac{\epsilon(C_{l'}(\eta)+a_{l'}(\eta)D_{l'}(\eta))+2a_{l'}(\eta)}{2C'} + \frac{1-a_l}{C'}\frac{1}{p}.$$

Therefore $(1/p, \alpha) \in \mathcal{A}_2$.

Finally we shall show that $(1/p, \alpha) \in \mathcal{A}_0$. There exists (x_0, y_0) such that $|g(x_0, y_0)| \geq k > 0$. Let

$$F_1(x) = \sum_{i=1}^{\infty} \frac{(\partial_x^i f)(x_0, y_0)}{i!}(x - x_0)^i$$

and

$$F_2(y) = \sum_{j=1}^{\infty} \frac{(\partial_y^j f)(x_0, y_0)}{j!}(y - y_0)^j.$$

We define $\psi_\lambda(x)$ and $\varphi_\lambda(y)$ by

$$\varphi_\lambda(y) = \begin{cases} e^{-i\lambda F_2(y)} & \text{if} \quad y_0 \leq y \leq y_0 + \lambda^{-1} \\ 0 & \text{otherwise,} \end{cases}$$

$$\psi_\lambda(x) = \begin{cases} e^{-i\lambda F_1(x)} & \text{if} \quad x_0 \leq x \leq x_0 + c_1 \\ 0 & \text{otherwise.} \end{cases}$$

By choosing a small number $c_1 > 0$ we have

$$|\lambda(f(x, y) - f(x_0, y_0) - F_1(x) - F_2(y))| \leq \pi/4.$$

Hence we have

$$\left| e^{-i\lambda f(x_0, y_0)} \int T_\lambda \varphi_\lambda(x) \psi_\lambda(x) dx \right| \geq C\lambda^{-1}$$

and

$$||f_\lambda||_{L^p} \sim \lambda^{-1/p} \quad \text{and} \quad ||g_\lambda||_{L^{p'}} \sim 1.$$

Therefore we have $\alpha \leq 1 - 1/p$. By exchanging the role of f_λ and g_λ we have $\alpha \leq 1/p$. This shows that $(1/p, \alpha) \in \mathcal{A}_0$. $\square$

3. Proof of theorem 1.3

Proof of Theorem 1.3. Recall that the quantities $a_l, a_{l'}(\eta), A_l, B_l, C_l, D_l$, etc. can be read off the generalized Newton diagrams of f''_{xy} and g. Let

$$f''_{xy}(x,y) = \prod_{\nu\in\Gamma_{f''_{xy}}} (y - r_\nu(x));\ g(x,y) = \prod_{\mu\in\Gamma_g} (y - s_\mu(x))$$

and suppose $0 < a_1 < a_2 < \cdots < a_l < a_{l+1} < \cdots < a_N$ is the combined set of leading exponents of r_ν's and s_μ's(arranged in increasing order of magnitude, without counting multiplicities). Without loss of generality, let $a_1 \geq 1$. We write

$$T^\lambda_{jk}\varphi(x) = \int_{\mathbb{R}} e^{i\lambda f(x,y)}|g(x,y)|^{\epsilon/2}\chi(x,y)\chi_j(x)\chi_k(y)\varphi(y)dy$$

where

$$\chi_i(z) = \begin{cases} 1 & \text{if } 2^{-i} \leq z \leq 2^{-i+1} \\ 0 & \text{otherwise.} \end{cases}$$

We consider four ranges of j, k,:

- $a_l j \ll k \ll a_{l+1} j$;
- $k \ll a_1 j$;
- $k \gg a_N$;
- $k \approx a_l j$,

where $A \ll B$, $A \gg B$, and $A \approx B$ mean that $A + C < B$, $A > B + C$, and $A - C < B < A + C$ respectively for some $C > 0$ which makes the following arguments hold true. Since the treatments of the first three cases are similar, we only consider two cases: $a_l j \ll k \ll a_{l+1} j$; $k \approx a_l j$.

Case 1: $a_l j \ll k \ll a_{l+1} j$

In this case

$$f''_{xy} \sim 2^{-A_l j} 2^{-B_l k}, \quad g \sim 2^{-C_l j} 2^{-D_l k}$$

on the support of $\chi_j(x)\chi_k(y)$. We have the following estimates of $||T_{jk}||$:

$$||T_{jk}|| \leq C(\lambda 2^{-A_l j - B_l k})^{-1/2} 2^{-\epsilon(C_l j + D_l k)/2}, \tag{3.1}$$

$$||T_{jk}|| \leq C 2^{-(j+k)/2} 2^{-\epsilon(C_l j + D_l k)/2}. \tag{3.2}$$

If we put $k = a_l j + r$ with $0 \ll r \ll (a_{l+1} - a_l)j$, we can rewrite (3.1) and (3.2) as

$$\begin{aligned} ||T_{jk}|| &\leq \min\left\{\lambda^{-1/2} 2^{j(A_l - \epsilon C_l)/2} 2^{k(B_l - \epsilon D_l)/2}, 2^{-j(1+\epsilon C_l)/2} 2^{-k(1+\epsilon D_l)/2}\right\} \\ &\leq \min\left\{\lambda^{-1/2} 2^{j[(A_l + a_l B_l) - \epsilon(C_l + a_l D_l)]/2} 2^{r(B_l - \epsilon D_l)/2}, \right. \\ &\qquad \left. 2^{-j[1 + a_l + \epsilon(C_l + a_l D_l)]/2} 2^{-r(1+\epsilon D_l)/2}\right\}. \end{aligned}$$

First we take into account of the case where

$$\lambda^{-1/2} 2^{j[(A_l + a_l B_l) - \epsilon(C_l + a_l D_l)]/2} 2^{r(B_l - \epsilon D_l)/2} \leq 2^{-j[1 + a_l + \epsilon(C_l + a_l D_l)]/2} 2^{-r(1+\epsilon D_l)/2},$$

which is equivalent to

$$2^{j(1 + a_l + A_l + a_l B_l)/2} \leq \lambda^{1/2} 2^{-r(1+B_l)/2}$$

i.e.,

$$2^{j/2} \leq \lambda^{\frac{1}{2(1 + a_l + A_l + a_l B_l)}} 2^{-\frac{r(1+B_l)}{2(1 + a_l + A_l + a_l B_l)}}. \tag{3.3}$$

By the choice of r we also have

$$2^{j/2} \geq 2^{\frac{r}{2(a_{l+1} - a_l)}}. \tag{3.4}$$

By combining (3.3) and (3.4) we obtain

$$2^{\frac{r}{2(a_{l+1} - a_l)}} \leq \lambda^{\frac{1}{2(1 + a_l + A_l + a_l B_l)}} 2^{-\frac{r(1+B_l)}{2(1 + a_l + A_l + a_l B_l)}},$$

which implies

$$2^{\frac{r}{2}} \leq \lambda^{\frac{1}{2}\cdot\frac{a_{l+1}-a_l}{1+a_{l+1}+A_l+a_{l+1}B_l}}. \tag{3.5}$$

Subcase 1: $A_l + a_l B_l \geq \epsilon(C_l + a_l D_l)$

In this case

$$\sum_j ||T_{jk}|| \leq \lambda^{-1/2}\lambda^{\frac{(A_l+a_lB_l)-\epsilon(C_l+a_lD_l)}{2(1+A_l+a_l+a_lB_l)}} 2^{\frac{r}{2}I}$$

where

$$I = (B_l - \epsilon D_l) - \frac{(1+B_l)[(A_l+a_lB_l)-\epsilon(C_l+a_lD_l)]}{1+a_l+A_l+a_lB_l}.$$

If $I < 0$, then

$$\sum_{(j,k);\, a_l j \ll k \ll a_{l+1} j} ||T_{jk}|| \leq \lambda^{-\frac{1}{2}\cdot\frac{1+a_l+\epsilon(C_l+a_lD_l)}{1+a_l+A_l+a_lB_l}}.$$

If $I \geq 0$, then

$$\begin{aligned}\sum_{(j,k);\, a_l j \ll k \ll a_{l+1} j} ||T_{jk}|| &\leq \lambda^{-\frac{1}{2}\cdot\frac{1+a_l+\epsilon(C_l+a_lD_l)}{1+a_l+A_l+a_lB_l}}\lambda^{\frac{1}{2}\cdot\frac{(a_{l+1}-a_l)I}{1+a_{l+1}+A_l+a_{l+1}B_l}} \\ &\leq \lambda^{-\frac{1}{2}[\frac{1+a_l+\epsilon(C_l+a_lD_l)}{1+a_l+A_l+a_lB_l}-\frac{(a_{l+1}-a_l)I}{1+a_{l+1}+A_l+a_{l+1}B_l}]}.\end{aligned}$$

We claim that

(3.6)
$$\frac{1+a_l+\epsilon(C_l+a_lD_l)}{1+a_l+A_l+a_lB_l} - \frac{(a_{l+1}-a_l)I}{1+a_{l+1}+A_l+a_{l+1}B_l} = \frac{1+a_{l+1}+\epsilon(C_l+a_{l+1}D_l)}{1+a_{l+1}+A_l+a_{l+1}B_l}.$$

By rewriting (3.6) we have to show

$$[1+a_l+\epsilon(C_l+a_lD_l)][1+a_{l+1}+A_l+a_{l+1}B_l] - (a_{l+1}-a_l)$$

$$\times [(B_l-\epsilon D_l)(1+a_l+A_l+a_lB_l) - (B_l+1)\{A_l+a_lB_l-\epsilon(C_l+a_lD_l)\}]$$

$$= [1+a_{l+1}+\epsilon(C_l+a_{l+1}D_l)][1+a_l+A_l+a_lB_l].$$

Now we take derivatives of the left and right hand sides with respect to a_{l+1}:

$$\begin{aligned}
\frac{d}{da_{l+1}}(\text{LHS}) &= (1+B_l)[1+a_l+\epsilon(C_l+a_lD_l)] \\
&\quad -[(B_l-\epsilon D_l)(1+a_l+A_l+a_lB_l)-(B_l+1)\{A_l+a_lB_l-\epsilon(C_l+a_lD_l)\}] \\
&= (1+B_l)[1+a_l+\epsilon(C_l+a_lD_l)+A_l+a_lB_l-\epsilon(C_l+a_lD_l)] \\
&\quad -(B_l-\epsilon D_l)(1+a_l+A_l+a_lB_l) \\
&= (1+\epsilon D_l)(1+a_l+A_l+a_lB_l), \\
\frac{d}{da_{l+1}}(\text{RHS}) &= (1+\epsilon D_l)(1+a_l+A_l+a_lB_l).
\end{aligned}$$

Also if $a_{l+1}=a_l$ then it is easy to see that the left hand side is same to the right hand side. Thus the claim is proved.

Subcase 2: $A_l+a_lB_l<\epsilon(C_l+a_lD_l)$

In this case

$$\begin{aligned}
\sum_j ||T_{jk}|| &\leq \lambda^{-\frac{1}{2}}2^{\frac{r}{2}\frac{A_l+a_lB_l-\epsilon(C_l+a_lD_l)}{a_{l+1}-a_l}}2^{\frac{r}{2}(B_l-\epsilon D_l)} \\
&\leq \lambda^{-\frac{1}{2}}2^{\frac{r}{2}\frac{A_l+a_{l+1}B_l-\epsilon(C_l+a_{l+1}D_l)}{a_{l+1}-a_l}}.
\end{aligned}$$

If $A_l+a_{l+1}B_l\geq\epsilon(C_l+a_{l+1}D_l)$, then (3.5) yields

$$\begin{aligned}
\sum_{(j,k);\ a_lj\ll k\ll a_{l+1}j} ||T_{jk}|| &\leq \lambda^{-\frac{1}{2}}\lambda^{\frac{1}{2}\cdot\frac{A_l+a_{l+1}B_l-\epsilon(C_l+a_{l+1}D_l)}{1+a_{l+1}+A_l+a_{l+1}B_l}} \\
&\leq \lambda^{-\frac{1}{2}\cdot\frac{1+a_{l+1}+\epsilon(C_l+a_{l+1}D_l)}{1+a_{l+1}+A_l+a_{l+1}B_l}} \\
&= \lambda^{-\frac{1}{2}\cdot\frac{1+a_{l+1}+\epsilon(C_{l+1}+a_{l+1}D_{l+1})}{1+a_{l+1}+A_{l+1}+a_{l+1}B_{l+1}}}
\end{aligned}$$

If $A_l + a_{l+1}B_l < \epsilon(C_l + a_{l+1}D_l)$, then

$$\sum_{(j,k);\, a_l j \ll k \ll a_{l+1} j} ||T_{jk}|| \leq \lambda^{-\frac{1}{2}}.$$

Now we consider the case where

$$2^{j/2} \geq \lambda^{\frac{1}{2(1+a_l+A_l+a_lB_l)}} 2^{-\frac{r(1+B_l)}{2(1+a_l+A_l+a_lB_l)}}. \tag{3.7}$$

(3.4) still holds in this case. We consider two cases:

$$\lambda^{\frac{1}{2(1+a_l+A_l+a_lB_l)}} 2^{-\frac{r(1+B_l)}{2(1+a_l+A_l+a_lB_l)}} \geq 2^{\frac{r}{2(a_{l+1}-a_l)}}; \tag{3.8}$$

$$\lambda^{\frac{1}{2(1+a_l+A_l+a_lB_l)}} 2^{-\frac{r(1+B_l)}{2(1+a_l+A_l+a_lB_l)}} < 2^{\frac{r}{2(a_{l+1}-a_l)}}. \tag{3.9}$$

We rewrite (3.8) as

$$2^{\frac{r}{2}} \leq \lambda^{\frac{a_{l+1}-a_l}{2(1+a_{l+1}+A_l+a_{l+1}B_{l+1})}}. \tag{3.10}$$

By using (3.7) we obtain

$$\sum_j ||T_{jk}|| \leq \lambda^{-\frac{1}{2}} \lambda^{\frac{(A_l+a_lB_l)-\epsilon(C_l+a_lD_l)}{2(1+A_l+a_l+a_lB_l)}} 2^{\frac{r}{2}I}.$$

If $I < 0$ then we have a convergent geometric series which we sum to obtain

$$\sum_{j,k} ||T_{jk}|| \leq \lambda^{-\frac{1}{2}\cdot\frac{1+a_l+\epsilon(C_l+a_lD_l)}{1+a_l+A_l+a_lB_l}}.$$

If $I \geq 0$ then (3.10) and (3.6) yield

$$\sum_{j,k} ||T_{jk}|| \leq \lambda^{-\frac{1}{2}\cdot\frac{1+a_{l+1}+\epsilon(C_l+a_{l+1}D_l)}{1+a_{l+1}+A_l+a_{l+1}B_l}}.$$

Now we rewrite (3.9) as

$$2^{\frac{r}{2}} > \lambda^{\frac{a_{l+1}-a_l}{2(1+a_{l+1}+A_l+a_{l+1}B_{l+1})}}. \tag{3.11}$$

In this case we use (3.4) to obtain

$$\sum_j ||T_{jk}|| \le 2^{-\frac{r}{2}\cdot\frac{1+a_l+\epsilon(C_l+a_lD_l)}{a_{l+1}-a_l}} 2^{-\frac{r}{2}\cdot(1+\epsilon D_l)}.$$

We then use (3.11) to get

$$\sum_{jk} ||T_{jk}|| \le \lambda^{-\frac{1+a_{l+1}+\epsilon(C_l+a_{l+1}D_l)}{2(1+a_{l+1}+A_l+a_{l+1}B_l)}},$$

which is the desired estimate. This completes the treatment of Case 1.

Case 2: $k \approx a_l j$

In this case we choose a real root $t(x)$ of $f''_{xy}(x,y) = 0$ or $g(x,y) = 0$ and put $y - t(x) \sim 2^{-m}$, $m \ge a_l j$. Let

$$a_l \le a_1(\eta) < a_2(\eta) < \cdots < a_k(\eta) < \cdots$$

be the set of leading exponents of $\{r_\nu(x)-t(x);\ \nu \in \Gamma_{f''_{xy}}\} \cup \{s_\mu(x)-t(x);\ \mu \in \Gamma_g\}$. If $a_k(\eta)j \ll m \ll a_{k+1}j$ then we have

$$f''_{xy} \sim 2^{-A_k(\eta)j}2^{-B_k(\eta)m}; \quad g \sim 2^{-C_k(\eta)j}2^{-D_k(\eta)m}.$$

We write

$$T^\lambda_{j,k,m}\varphi(x) = \int_{\mathbb{R}} e^{i\lambda f(x,y)}|g(x,y)|^{\epsilon/2}\chi(x,y)\varphi(y)\chi_j(x)\chi_k(y)\chi_m(y-t(x))dy.$$

The following estimates hold:

$$(3.12)\quad ||T^\lambda_{j,k,m}|| \le C(\lambda 2^{-(A_{l'}(\eta)j+B_{l'}(\eta)m)})^{-1/2}(2^{-(C_{l'}(\eta)j+D_{l'}(\eta)m)})^{\epsilon/2},$$

$$(3.13)\quad ||T^\lambda_{j,k,m}|| \le 2^{-m}2^{j(a_l-1)/2}(2^{-(C_{l'}(\eta)j+D_{l'}(\eta)m)})^{\epsilon/2}.$$

since $\Delta y \le 2^{-m}$ and $\Delta x \le 2^{-m}2^{a_l-1}$, where Δy is the maximal variation in y for a fixed x in the region under consideration and Δx is defined in a similar way. By putting $m = a_{l'}(\eta)j + r$ with $0 \ll r \ll (a_{l'+1}(\eta) - a_{l'}(\eta))j$, we have

$$\begin{aligned}||T^{\lambda}_{j,k,m}|| &\leq \min\Big\{\lambda^{-1/2}2^{j(A_{l'}(\eta)-\epsilon C_{l'}(\eta))/2}2^{m(B_{l'}(\eta)-\epsilon D_{l'}(\eta))/2},\\ &\quad 2^{j(a_l-1-\epsilon C_{l'}(\eta))/2}2^{-m(2+\epsilon D_{l'}(\eta)/2}\Big\}\\ &= \min\Big\{\lambda^{-1/2}2^{j[(A_{l'}(\eta)+a_{l'}(\eta)B_{l'}(\eta))-\epsilon(C_{l'}(\eta)+a_{l'}(\eta)D_{l'}(\eta))]/2}2^{r(B_{l'}(\eta)-\epsilon D_{l'}(\eta))/2},\\ &\quad 2^{-j[(1+2a_{l'}(\eta)-a_l)+\epsilon(C_{l'}(\eta)+a_{l'}(\eta)D_{l'}(\eta))]/2}2^{-r(2+\epsilon D_{l'}(\eta))/2}\Big\}.\end{aligned}$$

First we consider the case where

$$\begin{aligned}&\lambda^{-1/2}2^{j[(A_{l'}(\eta)+a_{l'}(\eta)B_{l'}(\eta))-\epsilon(C_{l'}(\eta)+a_{l'}(\eta)D_{l'}(\eta))]/2}2^{r(B_{l'}(\eta)-\epsilon D_{l'}(\eta))/2}\\ &\leq 2^{-j[(1+2a_{l'}(\eta)-a_l)+\epsilon(C_{l'}(\eta)+a_{l'}(\eta)D_{l'}(\eta))]/2}2^{-r(2+\epsilon D_{l'}(\eta))/2},\end{aligned}$$

that is,

$$\begin{aligned}2^{j/2} &\leq \lambda^{\frac{1}{2(A_{l'}(\eta)+a_{l'}(\eta)B_{l'}(\eta)+2a_{l'}(\eta)-a_l+1)}}\\ &\quad \times 2^{-\frac{r}{2}\cdot\frac{B_{l'}(\eta)+2}{A_{l'}(\eta)+a_{l'}(\eta)B_{l'}(\eta)+2_{l'}(\eta)-a_l+1}}.\end{aligned}$$

Also,

$$2^{j/2} \geq 2^{\frac{r}{2(a_{l'+1}(\eta)-a_{l'}(\eta))}}.$$

This implies

$$2^{\frac{r}{2}\left(\frac{1}{a_{l'+1}(\eta)-a_{l'}(\eta)}+\frac{B_{l'}(\eta)+2}{A_{l'}(\eta)+a_{l'}(\eta)B_{l'}(\eta)+2a_{l'}(\eta)-a_l+1}\right)} \leq \lambda^{\frac{1}{2(A_{l'}(\eta)+a_{l'}(\eta)B_{l'}(\eta)+2a_{l'}(\eta)-a_l+1)}}$$

which is equivalent to

$$2^{\frac{r}{2}(A_{l'}(\eta)+a_{l'+1}(\eta)B_{l'}(\eta)+2a_{l'+1}(\eta)-a_l+1)} \leq \lambda^{\frac{1}{2}(a_{l'+1}(\eta)-a_{l'}(\eta))},$$

i.e.,

$$2^{\frac{r}{2}} \leq \lambda^{\frac{1}{2}\frac{a_{l'+1}(\eta)-a_{l'}(\eta)}{A_{l'}(\eta)+a_{l'+1}(\eta)B_{l'}(\eta)+2a_{l'+1}(\eta)-a_l+1}}.$$

We therefore have

$$\begin{aligned}\sum_j ||T^\lambda_{j,k,m}|| &\leq \lambda^{-1/2}\lambda^{\frac{1}{2}\frac{(A_{l'}(\eta)+a_{l'}(\eta)B_{l'}(\eta))-\epsilon(C_{l'}(\eta)+a_{l'}(\eta)D_{l'}(\eta))}{A_{l'}(\eta)+a_{l'}(\eta)B_{l'}(\eta)+2a_{l'}(\eta)-a_l+1}} \\ &\quad \times 2^{-\frac{r}{2}\frac{(B_{l'}(\eta)+2)[(A_{l'}(\eta)+a_{l'}(\eta)B_{l'}(\eta))-\epsilon(C_{l'}(\eta)+a_{l'}(\eta)D_{l'}(\eta))]}{A_{l'}(\eta)+a_{l'}(\eta)B_{l'}(\eta)+2a_{l'}(\eta)-a_l+1}} \\ &\quad \times 2^{\frac{r}{2}(B_{l'}(\eta)-\epsilon D_{l'}(\eta))} \\ &\leq \lambda^{-\frac{1}{2}\frac{2a_{l'}(\eta)-a_l+1+\epsilon(C_{l'}(\eta)+a_{l'}(\eta)D_{l'}(\eta))}{2a_{l'}(\eta)-a_l+1+A_{l'}(\eta)+a_{l'}(\eta)B_{l'}(\eta)}}2^{\frac{r}{2}J},\end{aligned}$$

where

$$\begin{aligned}J &= (B_{l'}(\eta)-\epsilon D_{l'}(\eta)) \\ &\quad - \frac{(B_{l'}(\eta)+2)[(A_{l'}(\eta)+a_{l'}(\eta)B_{l'}(\eta))-\epsilon(C_{l'}(\eta)+a_{l'}(\eta)D_{l'}(\eta))]}{A_{l'+1}(\eta)+a_{l'}(\eta)B_{l'}(\eta)+2a_{l'}(\eta)-a_l+1}.\end{aligned}$$

If $J<0$, then

$$\sum_{j,k,m;\, a_{l'}(\eta)j\ll m\ll a_{l'+1}(\eta)j} ||T^\lambda_{j,k,m}|| \leq \lambda^{-\frac{1}{2}\frac{2a_{l'}(\eta)-a_l+1+\epsilon(C_{l'}(\eta)+a_{l'}(\eta)D_{l'}(\eta))}{2a_{l'}(\eta)-a_l+1+A_{l'}(\eta)+a_{l'}(\eta)B_{l'}(\eta)}}.$$

If $J\geq 0$, then

$$\begin{aligned}\sum_{j,k,m;\, a_{l'}(\eta)j\ll m\ll a_{l'+1}(\eta)j} ||T^\lambda_{j,k,m}|| &\leq \lambda^{-\frac{1}{2}\frac{2a_{l'}(\eta)-a_l+1+\epsilon(C_{l'}(\eta)+a_{l'}(\eta)D_{l'}(\eta))}{2a_{l'}(\eta)-a_l+1+A_{l'}(\eta)+a_{l'}(\eta)B_{l'}(\eta)}} \\ &\quad \times \lambda^{\frac{1}{2}\frac{(a_{l'+1}(\eta)-a_{l'}(\eta))J}{A_{l'}(\eta)+a_{l'+1}(\eta)B_{l'}(\eta)+2a_{l'+1}(\eta)-a_l+1}}.\end{aligned}$$

We claim that

$$\begin{aligned}&\frac{2a_{l'}(\eta)-a_l+1+\epsilon(C_{l'}(\eta)+a_{l'}(\eta)D_{l'}(\eta))}{2a_{l'}(\eta)-a_l+1+A_{l'}(\eta)+a_{l'}(\eta)B_{l'}(\eta)} - \frac{(a_{l'+1}(\eta)-a_{l'}(\eta))J}{2a_{l'+1}(\eta)-a_l+1+A_{l'}(\eta)+a_{l'+1}(\eta)B_{l'}(\eta)} \\ &= \frac{2a_{l'+1}(\eta)-a_l+1+\epsilon(C_{l'}(\eta)+a_{l'+1}(\eta)D_{l'}(\eta))}{2a_{l'+1}-a_l+1+A_{l'}(\eta)+a_{l'+1}(\eta)B_{l'}(\eta)},\end{aligned}$$

i.e.,

$$[2a_{l'}(\eta) - a_l + 1 + \epsilon(C_{l'}(\eta) + a_{l'}(\eta)D_{l'}(\eta))][2a_{l'+1}(\eta) - a_l + 1 + A_{l'}(\eta) + a_{l'+1}(\eta)B_{l'}(\eta)]$$
$$- (a_{l'+1}(\eta) - a_{l'}(\eta))[(B_{l'}(\eta) - \epsilon D_{l'}(\eta)(A_{l'}(\eta) + a_{l'}(\eta)B_{l'}(\eta) + 2a_{l'}(\eta) - a_l + 1)$$
$$- (B_{l'}(\eta) + 2)\{(A_{l'}(\eta) + a_{l'}(\eta)B_{l'}(\eta) - \epsilon(C_{l'}(\eta) + a_{l'}(\eta)D_{l'}(\eta))\}]$$
$$= (A_{l'}(\eta) + a_{l'}(\eta)B_{l'}(\eta) + 2a_{l'}(\eta) - a_l + 1)[2a_{l'+1}(\eta) - a_l + 1 + \epsilon(C_{l'}(\eta) + a_{l'}(\eta)D_{l'}(\eta))].$$

The claim is proved using the same techniques as before.

Now we have to treat the case where

$$2^{j/2} > \lambda^{\frac{1}{2(A_{l'}(\eta)+a_{l'}(\eta)B_{l'}(\eta)+2a_{l'}(\eta)-a_l+1)}} \times 2^{-\frac{r}{2}\cdot\frac{B_{l'}(\eta)+2}{A_{l'}(\eta)+a_{l'}(\eta)B_{l'}(\eta)+2_{l'}(\eta)-a_l+1}}.$$

Since we can directly apply an earlier argument to handle this case, we omit the detail here.

If $m \approx a_{l'}(\eta)j$, then there exists $\tilde{t} \in$ such that $y - \tilde{t}(x)$ is "small". Put $y - \tilde{t}(x) \sim 2^{-p}$ and repeat the same arguments as before. Then we conclude

$$||T_\lambda|| \le C\lambda^{-\delta/2}$$

where

$$\delta = \min\left(\frac{1}{2}, \frac{1}{2}\cdot\frac{1 + a_l + \epsilon(C_l + a_lD_l)}{1 + a_l + A_l + a_lB_l}, \frac{1}{2}\cdot\frac{1 + 2a_{l'}(\eta) - a_l + \epsilon(C_{l'}(\eta) + a_{l'}(\eta)D_{l'}(\eta))}{1 + 2a_{l'}(\eta) - a_l + (A_{l'}(\eta) + a_{l'}(\eta)B_{l'}(\eta))}\right).$$

□

4. Proof of theorem 1.4

In this section we will prove theorem 1.4. We construct an analytic family of operators T_λ^β so that when $\mathrm{Re}(\beta) = 1/2$, T_λ^β is a damped oscillatory integral operator of the form

$$T_\lambda^{1/2}\varphi(x) = \int e^{i\lambda f(x,y)} |f''_{xy}(x,y)|^{1/2}\chi(x,y)\varphi(y)dy,$$

whose L^2 decay estimate we know of. When $\mathrm{Re}(\beta) = -\alpha/(1-2\alpha)$, we shall prove T_λ^β is bounded on $L^{\frac{p(1-2\alpha)}{1-p\alpha}}$, which yields theorem 1.4 by complex interpolation in [StW].

Proof of Theorem 1.4. We consider an analytic family of operators

$$T_\lambda^\beta\varphi(x) = \int e^{i\lambda f(x,y)} |g(x,y)|^{\epsilon(1/2-\beta)} |f''_{xy}(x,y)|^{\beta}\chi(x,y)\varphi(y)dy. \tag{4.1}$$

We note that $T_\lambda^0 = T_\lambda$ and that if $\mathrm{Re}(\beta) = 1/2$ then we have

Theorem 4.1 ([PSt4])**.**

$$||T_\lambda^\beta||_{L^2\to L^2} = O(\lambda^{-1/2}).$$

When $\mathrm{Re}(\beta) = -\alpha/(1-2\alpha)$, T_λ^β is a form of fractional integration and we want to obtain estimate without any decay rate. To do this we shall use the following lemma.

Lemma 4.2. *If $K(x,y) \geq 0$ be the kernel of an operator T and $K(x,y)$ satisfies the following,*

$$\int K(x,y)y^{-\frac{1}{p}}dy \leq Cx^{-\frac{1}{p}}, \quad \int K(x,y)x^{-\frac{1}{q}}dx \leq Cy^{-\frac{1}{q}},$$

where $1/p + 1/q = 1$, then

$$T\varphi(x) = \int K(x,y)\varphi(y)dy$$

is bounded in L^p.

Proof of Lemma 4.2. For $\varphi \in L^p$ and $\psi \in L^q$ ($\frac{1}{p}+\frac{1}{q}=1$) with $||\varphi||_p = ||\psi||_q = 1$, we have

$$\begin{aligned} |\varphi(y)\psi(x)| &= |\varphi(y)x^{-\frac{1}{pq}}y^{\frac{1}{pq}}\psi(x)y^{-\frac{1}{pq}}x^{\frac{1}{pq}}| \\ &\leq \frac{1}{p}|\varphi(y)|^p x^{-\frac{1}{q}}y^{\frac{1}{q}} + \frac{1}{q}|\psi(x)|^q y^{-\frac{1}{p}}x^{\frac{1}{p}}. \end{aligned}$$

Therefore, we have

$$\begin{aligned} &\left|\int\int K(x,y)\varphi(y)\psi(x)dydx\right| \\ \leq\ & \int\int K(x,y)\frac{1}{p}|\varphi(y)|^p x^{-\frac{1}{q}}y^{\frac{1}{q}}dydx + \int\int K(x,y)\frac{1}{q}|\psi(x)|^q y^{-\frac{1}{p}}x^{\frac{1}{p}}dydx \\ \leq\ & C/p + C/q. \end{aligned}$$

This completes the proof. □

Now we shall prove the following lemma.

Lemma 4.3. *If* $(1/p,\alpha) \in \mathrm{int}(\mathcal{A})$, *then* $T_\lambda^{-\alpha/(1-2\alpha)}$ *is bounded on* $L^{\frac{p(1-2\alpha)}{1-p\alpha}}$ *with the operator norm* $O(1)$.

Proof of Lemma 4.3. Since the oscillation does not play any role, it suffices to obtain $L^{\frac{p(1-2\alpha)}{1-p\alpha}}$ boundedness of the operator

$$D\varphi(x) = \int |g(x,y)|^{\frac{\epsilon}{2(1-2\alpha)}}|f''_{xy}(x,y)|^{\frac{-\alpha}{1-2\alpha}}\chi(x,y)\varphi(y)dy.$$

Let

$$K(x,y) = |g(x,y)|^{\frac{\epsilon}{2(1-2\alpha)}}|f''_{xy}(x,y)|^{\frac{-\alpha}{1-2\alpha}}.$$

By lemma 4.2, it suffices to show that

$$\int_I K(x,y)\frac{1}{y^{\frac{1-p\alpha}{p(1-2\alpha)}}}dy \leq \frac{C}{x^{\frac{1-p\alpha}{p(1-2\alpha)}}} \tag{4.2}$$

and

$$\int_I K(x,y)\frac{1}{x^{\frac{p-1-p\alpha}{p(1-2\alpha)}}}dx \le \frac{C}{y^{\frac{p-1-p\alpha}{p(1-2\alpha)}}}, \tag{4.3}$$

where $I=[-|I|,|I|]$ with a sufficiently small $|I|$. Since the argument to prove (4.3) is pararell to the argument for (4.2), we shall only show (4.2). The proof can be divided into finite steps and we shall here show the first two steps. To complete the proof we can repeat the same argument.

Step I Considering each quadrant separately, we may assume that $x>0$, $y>0$ and $I=[0,|I|]$. After reindexing if necessary, we may assume without loss of generality that we can find $c_l>0$, $d_l>0$, and $C_l>0$ such that $c_l<d_l<C_l$, $|r_l(x)| = d_l x^{a_l}+o(x^{a_l})$, and $|s_l(x)| = d_l x^{a_l}+o(x^{a_l})$. We split I into several subintervals: $0\le y\le c_n x^{a_n}$, $c_l x^{a_l}\le y\le C_l x^{a_l}$, $C_{l+1}x^{a_{l+1}}\le y\le c_l x^{a_l}$, and $C_1 x^{a_1}\le y\le |I|$ and separately treat the each case.

Case 1. $0\le y\le c_n x^{a_n}$

If $0\le y\le c_n x^{a_n}$, then

$$g(x,y)\sim x^{C_n}y^{D_n}, \quad \text{and} \quad f''_{xy}(x,y)\sim x^{A_n}y^{B_n}.$$

Since

$$\alpha < \frac{\epsilon D_n+2}{2(B_n+1)} - \frac{1}{B_n+1}\frac{1}{p}, \tag{4.4}$$

$$\frac{\epsilon D_n - 2\alpha B_n}{2(1-2\alpha)} - \frac{1-p\alpha}{p(1-2\alpha)} > -1.$$

Consequently,

$$\begin{aligned}\int_0^{c_n x^{a_n}} K(x,y)\frac{1}{y^{\frac{1-p\alpha}{p(1-2\alpha)}}}dy &\sim \int_0^{c_n x^{a_n}} x^{\frac{\epsilon C_n - 2\alpha A_n}{2(1-2\alpha)}} y^{\frac{\epsilon D_n - 2\alpha B_n}{2(1-2\alpha)}} \frac{1}{y^{\frac{1-p\alpha}{p(1-2\alpha)}}}dy \\ &\le x^{\frac{1}{1-2\alpha}[\frac{\epsilon(C_n+\alpha D_n)}{2} - \alpha A_n - \alpha a_n B_n - \frac{a_n}{p} - \alpha a_n] + a_n}.\end{aligned}$$

The desired estimate is followed by

$$\begin{aligned}
&\frac{1}{1-2\alpha}[\frac{\epsilon(C_n+\alpha D_n)}{2}-\alpha A_n-\alpha a_n B_n-\frac{a_n}{p}-\alpha a_n]+a_n+\frac{1-p\alpha}{p(1-2\alpha)}\\
=&\ \frac{1}{1-2\alpha}\left[\frac{\epsilon(C_n+a_nD_n)+2a_n}{2}+\frac{1-a_n}{p}-\alpha(1+a_n+A_n+a_nB_n)\right]\\
\geq&\ 0.
\end{aligned}$$

Case 2. $C_l x^{a_l}\leq y\leq c_{l+1}x^{a_{l+1}}$

If $C_l x^{a_l}\leq y\leq c_{l+1}x^{a_{l+1}}$,

$$g(x,y)\sim x^{C_l}y^{D_l},\quad \text{and}\quad f''_{xy}(x,y)\sim x^{A_l}y^{B_l}.$$

Therefore

$$\begin{aligned}
&\int_{C_{l+1}x^{a_{l+1}}}^{c_l x^{a_l}} K(x,y)\frac{1}{y^{\frac{1-p\alpha}{p(1-2\alpha)}}}dy\\
\sim&\ \int_{C_{l+1}x^{a_{l+1}}}^{c_l x^{a_l}} x^{\frac{\epsilon C_l}{2(1-2\alpha)}-\frac{A_l\alpha}{1-2\alpha}}y^{\frac{\epsilon D_l}{2(1-2\alpha)}-\frac{B_l\alpha}{1-2\alpha}-\frac{1-p\alpha}{p(1-2\alpha)}}dy\\
\leq&\ x^{\frac{\epsilon C_l}{2(1-2\alpha)}-\frac{A_l\alpha}{1-2\alpha}}x^{\frac{\epsilon a_l D_l}{2(1-2\alpha)}-\frac{a_lB_l\alpha}{1-2\alpha}-\frac{(1-p\alpha)a_l}{p(1-2\alpha)}+a_l}\\
&+x^{\frac{\epsilon C_l}{2(1-2\alpha)}-\frac{A_l\alpha}{1-2\alpha}}x^{\frac{\epsilon a_{l+1} D_l}{2(1-2\alpha)}-\frac{a_{l+1}B_l\alpha}{1-2\alpha}-\frac{(1-p\alpha)a_{l+1}}{p(1-2\alpha)}+a_{l+1}}\\
\leq&\ x^{\frac{\epsilon C_l}{2(1-2\alpha)}-\frac{A_l\alpha}{1-2\alpha}}x^{\frac{\epsilon a_l D_l}{2(1-2\alpha)}-\frac{a_lB_l\alpha}{1-2\alpha}-\frac{(1-p\alpha)a_l}{p(1-2\alpha)}+a_l}\\
&+x^{\frac{\epsilon C_{l+1}}{2(1-2\alpha)}-\frac{A_{l+1}\alpha}{1-2\alpha}}x^{\frac{\epsilon a_{l+1} D_{l+1}}{2(1-2\alpha)}-\frac{a_{l+1}B_{l+1}\alpha}{1-2\alpha}-\frac{(1-p\alpha)a_{l+1}}{p(1-2\alpha)}+a_{l+1}}
\end{aligned}$$

where we use the following for the last inequality

(4.5) $\quad C_l+a_{l+1}D_l=C_{l+1}+a_{l+1}D_{l+1}\quad$ and $\quad A_l+a_{l+1}B_l=A_{l+1}+a_{l+1}B_{l+1}.$

Case 3. $C_1x^{a_1}\leq y\leq |I|$

If $C_1x^{a_1}\leq y\leq |I|$,

$$g(x,y)\sim x^{C_0}y^{D_0},\quad \text{and}\quad f''_{xy}(x,y)\sim x^{A_0}y^{B_0}.$$

By using (4.5) again, we obtain

$$\begin{aligned}\int_{C_1x^{a_1}}^{|I|} K(x,y)\frac{1}{y^{\frac{1-p\alpha}{p(1-2\alpha)}}}dy &\sim x^{\frac{\epsilon C_0}{2(1-2\alpha)}-\frac{\alpha A_0}{1-2\alpha}}\int_{C_1x^{a_1}}^{|I|} y^{\frac{\epsilon D_0}{2(1-2\alpha)}-\frac{\alpha B_0}{1-2\alpha}}dy \\ &\leq x^{\frac{\epsilon C_0}{2(1-2\alpha)}-\frac{\alpha A_0}{1-2\alpha}}(1+x^{\frac{\epsilon D_0 a_1}{2(1-2\alpha)}-\frac{\alpha B_0 a_1}{1-2\alpha}+a_1}) \\ &\leq x^{\frac{\epsilon C_0}{2(1-2\alpha)}-\frac{\alpha A_0}{1-2\alpha}}+x^{\frac{\epsilon(C_0+a_1D_0)}{2(1-2\alpha)}-\frac{\alpha(A_0+a_1B_0)}{1-2\alpha}+a_1} \\ &= x^{\frac{\epsilon C_0}{2(1-2\alpha)}-\frac{\alpha A_0}{1-2\alpha}}+x^{\frac{\epsilon(C_1+a_1D_2)}{2(1-2\alpha)}-\frac{\alpha(A_1+a_1B_1)}{1-2\alpha}+a_1}.\end{aligned}$$

Since

$$\alpha \leq \frac{\epsilon C_0}{2(1+A_0)}+\frac{1}{1+A_0}\frac{1}{p}$$

and

$$\alpha \leq \frac{\epsilon(C_1+a_1D_1)+a_1}{2(1+a_1+A_1+a_1B_1)}+\frac{1-a_1}{1+a_1+A_1+a_1B_1}\frac{1}{p},$$

$$\int_{C_1x^{a_1}}^{|I|} K(x,y)\frac{1}{y^{\frac{1-p\alpha}{p(1-2\alpha)}}}dy \leq x^{-\frac{1-p\alpha}{p(1-2\alpha)}},$$

which finishes the treatment of Case 3.

Case 4. $c_lx^{a_l} \leq y \leq C_lx^{a_l}$

If $c_lx^{a_l} \leq y \leq C_lx^{a_l}$,

$$\begin{aligned}g(x,y) &\sim x^{C_{l-1}}y^{D_l}\prod_{c_lx^{a_l}\leq|s_i(x)|\leq C_lx^{a_l}}|y-s_i(x)|, \\ f''_{xy}(x,y) &\sim x^{A_{l-1}}y^{B_l}\prod_{c_lx^{a_l}\leq|r_i(x)|\leq C_lx^{a_l}}|y-r_i(x)|.\end{aligned}$$

To treat this case we need finer decomposition of the domain of integration so we start the second step.

Step II To do this we introduce the following notation:

$$S_l^\alpha = \{r_i(x)|r_i(x) = c_l^\alpha x^{a_l}+o(x^{a_l})\}.$$

We assumed that for all $r_j(x)$ and $s_j(x)$ satisfying

$$c_l x^{a_l} < |r_j(x)|,\ |s_j(x)| < C_l x^{a_l},$$

$|r_j(x)|$ and $|s_j(x)|$ have the same leading term $d_l x^{a_l}$, that is,

$$|r_j(x)| = d_l x^{a_l} + o(x^{a_l}) \quad \text{and} \quad |s_j(x)| = d_l x^{a_l} + o(x^{a_l}).$$

If we set $r_j(x) = c_l^{\alpha} x^{a_l} + o(x^{a_l})$, we have three possible cases: (i) $\mathrm{Im}(c_l^{\alpha}) \neq 0$, (ii) $c_l^{\alpha} < 0$, and (iii) $c_l^{\alpha} > 0$. In (i) and (ii), we have

$$|y - r_j(x)| \sim x^{a_l}$$

if y is in the range $\{c_l x^{a_l} < y < C_l x^{a_l}\}$. Hence we may assume that $c_l^{\alpha} = d_l > 0$. Now we define a coordinate transformation η_1 so that

$$\eta_1(x, y) = (x, y + c_l^{\alpha} x^{a_l}).$$

If we rewrite the integral in terms of y_1, we have

$$\begin{aligned} \int_{c_l x^{a_l}}^{C_l x^{a_l}} K(x,y) \frac{1}{y^{\frac{1-p\alpha}{p(1-2\alpha)}}} dy &\leq x^{-\frac{(1-p\alpha)a_l}{p(1-2\alpha)}} \int_{-Cx^{a_l}}^{Cx^{a_l}} K(x, y_1 + c_l^{\alpha} x^{a_l}) dy_1 \\ &= x^{-\frac{(1-p\alpha)a_l}{p(1-2\alpha)}} \int_{-Cx^{a_l}}^{0} K(x, y_1 + c_l^{\alpha} x^{a_l}) dy_1 \\ &\quad + x^{-\frac{(1-p\alpha)a_l}{p(1-2\alpha)}} \int_{0}^{Cx^{a_l}} K(x, y_1 + c_l^{\alpha} x^{a_l}) dy_1 \\ &= I_{l,-} + I_{l,+}. \end{aligned}$$

Since the treatment of $I_{l,+}$ is similar to that of $I_{l,-}$, we only treat $I_{l,+}$. To do this we may assume that we can find $c_{l,l'}$, $d_{l,l'}$, and $C_{l,l'}$ such that $0 < c_{l,l'} < d_{l,l'} < C_{l,l'}$,

$$|r_l(x) - c_l^{\alpha} x^{a_l}| = d_{l,l'} x^{a_{l'}(\eta_1)} + o(x^{a_{l'}(\eta_1)}),$$

and

$$|s_l(x) - c_l^\alpha x^{a_l}| = d_{l,l'} x^{a_{l'}(\eta_1)} + o(x^{a_{l'}(\eta_1)}).$$

We decompose the region $\{(x,y) : 0 \le y \le Cx^{a_l}\}$ into several subregions: $0 \le y \le c_{l,n_1} x^{a_{n_1}(\eta_1)}$, $C_{l,1} x^{a_1(\eta_1)} \le y \le Cx^{a_l}$, $C_{l,l'+1} x^{a_{l'+1}(\eta_1)} \le y \le c_{l,l'} x^{a_{l'}(\eta_1)}$, and $c_{l,l'} x^{a_{l'}(\eta_1)} \le y \le C_{l,l'} x^{a_{l'}(\eta_1)}$.

Case 1. $0 \le y \le c_{l,n_1} x^{a_{n_1}(\eta_1)}$

In this case we have

$$\begin{aligned} &\int_0^{c_{l,n_1} x^{a_{n_1}(\eta_1)}} K(x, y + c_l^\alpha x^{a_l}) dy_1 \\ \sim \;& \int_0^{c_{l,n_1} x^{a_{n_1}(\eta_1)}} x^{\frac{\epsilon}{2(1-2\alpha)}(C_{n_1}(\eta_1) + a_{n_1}(\eta_1) D_{n_1}(\eta_1))} x^{\frac{-\alpha}{1-2\alpha}(A_{n_1}(\eta_1) + a_{n_1}(\eta_1) B_{n_1}(\eta_1))} dy_1 \\ = \;& x^{a_{n_1}(\eta_1)} x^{\frac{\epsilon}{2(1-2\alpha)}(C_{n_1}(\eta_1) + a_{n_1}(\eta_1) D_{n_1}(\eta_1))} x^{\frac{-\alpha}{1-2\alpha}(A_{n_1}(\eta_1) + a_{n_1}(\eta_1) B_{n_1}(\eta_1))}. \end{aligned}$$

To show the desired inequality we have to show

$$\begin{aligned} -\frac{(1-p\alpha)a_l}{p(1-2\alpha)} + a_{n_1}(\eta_1) + \frac{\epsilon}{2(1-2\alpha)}(C_{n_1}(\eta_1) + a_{n_1}(\eta_1) D_{n_1}(\eta_1)) \\ - \frac{\alpha}{1-2\alpha}(A_{n_1}(\eta_1) + a_{n_1}(\eta_1) B_{n_1}(\eta_1)) + \frac{1-p\alpha}{p(1-2\alpha)} \ge 0. \end{aligned}$$

To show this we factor out $\frac{1}{1-2\alpha}$ and simplify the left-hand side to get

$$\frac{1}{1-2\alpha}\left((1-a_l)\frac{1}{p} + \frac{\epsilon}{2}(C_{n_1}(\eta_1) + a_{n_1}(\eta_1) D_{n_1}(\eta_1) + 2a_{n_1}(\eta_1)) - \alpha C'\right) \tag{4.6}$$

where $C' = A_{n_1}(\eta_1) + a_{n_1}(\eta_1) B_{n_1}(\eta_1) + 2a_{n_1}(\eta_1) - a_l$. Now it is easy to see that (4.6) is nonnegative because of the assumption that $(1/p, \alpha) \in \mathcal{A}$.

Case 2. $C_{l,l'+1}x^{a_{l'+1}(\eta_1)} \le y \le c_{l,l'}x^{a_{l'}(\eta_1)}$

$$\begin{aligned}
&\int_{C_{l,l'+1}x^{a_{l'+1}(\eta_1)}}^{c_{l,l'}x^{a_{l'}(\eta_1)}} K(x, y_1 + c_l^\alpha x^{a_l})dy_1 \\
\sim \quad & \int_{C_{l,l'+1}x^{a_{l'+1}(\eta_1)}}^{c_{l,l'}x^{a_{l'}(\eta_1)}} x^{\frac{\epsilon C_{l'}(\eta_1)}{2(1-2\alpha)} - \frac{A_{l'}(\eta_1)\alpha}{1-2\alpha}} y^{\frac{\epsilon D_{l'}(\eta_1)}{2(1-2\alpha)} - \frac{B_{l'}(\eta_1)\alpha}{1-2\alpha}} dy_1 \\
\le \quad & x^{a_{l'}(\eta_1)} x^{\frac{\epsilon}{2(1-2\alpha)}(C_{l'}(\eta_1) + a_{l'}(\eta_1)D_{l'}(\eta_1))} x^{\frac{-\alpha}{1-2\alpha}(A_{l'}(\eta_1) + a_{l'}(\eta_1)B_{l'}(\eta_1))} \\
& + x^{a_{l'+1}(\eta_1)} x^{\frac{\epsilon}{2(1-2\alpha)}(C_{l'}(\eta_1) + a_{l'+1}(\eta_1)D_{l'}(\eta_1))} x^{\frac{-\alpha}{1-2\alpha}(A_{l'}(\eta_1) + a_{l'+1}(\eta_1)B_{l'}(\eta_1))} \\
\le \quad & x^{a_{l'}(\eta_1)} x^{\frac{\epsilon}{2(1-2\alpha)}(C_{l'}(\eta_1) + a_{l'}(\eta_1)D_{l'}(\eta_1))} x^{\frac{-\alpha}{1-2\alpha}(A_{l'}(\eta_1) + a_{l'}(\eta_1)B_{l'}(\eta_1))} \\
& + x^{a_{l'+1}(\eta_1)} x^{\frac{\epsilon}{2(1-2\alpha)}(C_{l'+1}(\eta_1) + a_{l'+1}(\eta_1)D_{l'+1}(\eta_1))} x^{\frac{-\alpha}{1-2\alpha}(A_{l'+1}(\eta_1) + a_{l'+1}(\eta_1)B_{l'+1}(\eta_1))}
\end{aligned}$$

where we use the following identities for the last inequality

$$\begin{aligned}
A_{l'}(\eta_1) + a_{l'+1}(\eta_1)B_{l'}(\eta_1) &= A_{l'+1}(\eta_1) + a_{l'+1}(\eta_1)B_{l'+1}(\eta_1) \\
C_{l'}(\eta_1) + a_{l'+1}(\eta_1)D_{l'}(\eta_1) &= C_{l'+1}(\eta_1) + a_{l'+1}(\eta_1)D_{l'+1}(\eta_1).
\end{aligned}$$

Case 3. $C_{l,1}x^{a_1(\eta_1)} \le y \le Cx^{a_l}$

$$\begin{aligned}
&\int_{C_{l,1}x^{a_1(\eta_1)}}^{Cx^{a_l}} K(x, y_1 + c_l^\alpha x^{a_l})dy_1 \\
\le \quad & x^{\frac{\epsilon C_l}{2(1-2\alpha)} - \frac{A_l\alpha}{1-2\alpha}} x^{\frac{\epsilon a_l D_l}{2(1-2\alpha)} - \frac{a_l B_l \alpha}{1-2\alpha} + a_l} \\
& + x^{a_1(\eta_1)} x^{\frac{\epsilon}{2(1-2\alpha)}(C_1(\eta_1) + a_1(\eta_1)D_1(\eta_1))} x^{\frac{-\alpha}{1-2\alpha}(A_1(\eta_1) + a_1(\eta_1)B_1(\eta_1))}
\end{aligned}$$

Case 4. $c_{l,l'}x^{a_{l'}(\eta_1)} \le y \le C_{l,l'}x^{a_{l'}(\eta_1)}$

It remains to show

$$x^{-\frac{(1-p\alpha)a_l}{p(1-2\alpha)}} \int_{c_{l,l'}x^{a_{l'}(\eta_1)}}^{C_{l,l'}x^{a_{l'}(\eta_1)}} K(x, y_1 + c_l^\alpha x^{a_l})dy_1 \le Cx^{\frac{-(1-p\alpha)}{p(1-2\alpha)}}.$$

To treat this case we start the third step which has the same argument with the second step. We repeat the same argument until we completely resolve the roots of f''_{xy} and g, that is, until there is only one root in the range of the integral. If we have only one root $r(x)$ in the range of the integral and if the root is a real root, we have to integrate $|y-r(x)|^{-(2\alpha B_{n(\eta)}(\eta)-\epsilon D_{n(\eta)}(\eta))/2(1-2\alpha)}$ with respect to y near $r(x)$, where η is a coordinate change defined by $\eta(x,y)=(x,y-r(x))$ and $n(\eta)$ is the largest index of $a_{l'}(\eta)$. The convergence of the integration is guarranteed because by using (1.1) we have

$$\alpha < \frac{\epsilon D_{n(\eta)}(\eta)) + 2}{2(B_{n(\eta)}(\eta)+2)} \tag{4.7}$$

and (4.7) implies

$$\frac{2\alpha B_{n(\eta)}(\eta) - \epsilon D_{n(\eta)}(\eta)}{2(1-2\alpha)} < 1.$$

We can easily see that we have the desired estimates for all integrals which will occur in each step. □

To finish the proof of Theorem 1.4 we interpolate Lemma 4.3 with Theorem 4.1. □

Remark 4.4. 1. In the proof of Theorem 1.4, we use the strict inequalities at two places (4.4) and (4.7). When we prove (4.3), we have to use one more strict inequality

$$\alpha < \frac{\epsilon C_0}{2(1+A_0)} + \frac{1}{1+A_0}\frac{1}{p}. \tag{4.8}$$

Therefore, Theorem 1.3. can be extended to the boundary of $\mathcal{A}$ when $(1/p,\alpha)$ is not on any of a line which bounds the region in (4.4), (4.7) or (4.8). It would be interesting to obtain L^p decay estimates when $(1/p,\alpha)$ is on one of these lines.

2. Let δ_1 and δ_2 be the weighted Newton distance and the optimal decay rate, respectively. We give an example showing that in general the optimal decay rate for L^2 operator norm of T_λ can be smaller than the weighted Newton distance which has been introduced in [Pr]. We take f and g such that

$$\begin{aligned} f''_{xy}(x,y) &= (y-x^N)^{R_1}(y-x^N-x^{kN})^{M_1} \\ g(x,y) &= (y-x^N-x^{2N})^{R_2}. \end{aligned}$$

Without any change of variable, we have

$$a_1 = N,\ A_1 = N(R_1+M_1),\ B_1 = 0,\ C_1 = NR_2, \text{ and } D_1 = 0.$$

One can check that

$$\delta_1 = \frac{1+N+\epsilon N R_2}{1+N+N(R_1+M_1)}.$$

By using the change of variables $\eta : (x,y) \mapsto (x, y-x^N)$, we have

$$a_2(\eta) = kN,\ A_2 = kNM_1,\ B_2 = R,\ C_2 = 2NR_2, \text{ and } D_2 = 0.$$

We then have

$$\delta_2 = \frac{1+2kN-N+\epsilon(2NR_2)}{1+2kN-N+kN(M_1+R_1)}.$$

Given N there exists k such that

$$\delta_2 \sim \frac{2N}{2N+N(M_1+R_1)} = \frac{2}{2+M_1+R_1}.$$

For large N, we have

$$\delta_1 \sim \frac{1+\epsilon R_2}{1+R_1+M_1}.$$

Now choosing ϵ and R_2 so that $\epsilon R_2 > 1$, we get $\delta_2 < \delta_1$.

3. Theorem 1.3 may be expressed in one of the following two equivalent forms. Fix $\lambda \geq \lambda_0$ for some λ_0, sufficiently large. A set of the form

$$B = \{(x,y) \in \text{supp}\chi \mid a \leq x \leq b,\ c \leq y \leq d\}$$

is defined to be a "testing box" if there exist functions $F_1, F_2 : \mathbb{R} \to \mathbb{R}$ depending on B satisfying

$$\sup_{(x,y)\in B} |\lambda(f(x,y) - F_1(x) - F_2(y))| < \frac{\pi}{4}.$$

Let $\mathfrak{F}$ denote the class of all testing boxes.

Statement 1. *For large values of* λ*,*

$$||T_\lambda||_{L^2\to L^2} \sim \max\left\{\sup_{B\in\mathfrak{F}} |B|^{\frac{1}{2}} \inf_{(x,y)\in B} |g(x,y)|^{\epsilon/2}, \lambda^{-1/2}\right\}.$$

It suffices to consider a subfamily of $\mathfrak{F}$. Let us make the following definition:

Definition 4.5. A vector (a,b,c,d) with $b \geq a \geq 1$, $c, d > 0$ is called an *admissible tuple* if there exists a Puiseux series q with leading exponent a and constants k_i, r_i, m_i, $i = 1, 2$ such that for all λ sufficiently large

(1)

$$\begin{aligned} B &= \{(x,y); \lambda^{-1/c} + r_1\lambda^{-\frac{b-a+1}{c}} \leq x \leq \lambda^{-1/c} + r_2\lambda^{-\frac{b-a+1}{c}}, \\ &\qquad q(\lambda^{-1/c}) + m_1\lambda^{-b/c} \leq q(\lambda^{-1/c}) + m_2\lambda^{-b/c}\} \end{aligned}$$

is a "testing box", and

(2)

$$k_1\lambda^{-d/c} \leq \sup_{(x,y)\in B} |g(x,y)| \leq k_2\lambda^{-d/c}.$$

Let $\mathfrak{N}$ denote the collection of all admissible tuples.

Statement 2. *For large values of λ,*

$$||T_\lambda||_{L^2\to L^2} \sim \lambda^{-\delta/2}$$

where

$$\delta = \min\left\{\min_{(a,b,c,d)\in\mathfrak{N}} \frac{1+2b-a+\epsilon d}{c}, 1\right\}.$$

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