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CONVERGENCE OF TWO-DIMENSIONAL WEIGHTED INTEGRALS

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ABSTRACT. A two-dimensional weighted integral in $\mathbb{R}^2$ is proposed as a tool for analyzing higher-dimensional unweighted integrals, and a necessary and sufficient condition for the finiteness of the weighted integral is obtained.

1. INTRODUCTION

In this paper, we consider weighted integrals in $\mathbb{R}^2$ of the form

$$\int_B \frac{|g(x,y)|^\epsilon}{|f(x,y)|^\delta}\, dy\, dx, \tag{1.1}$$

where f and g are real-analytic (possibly complex-valued) functions on $\mathbb{R}^2$, ϵ, δ are positive numbers and B is a small ball centered at the origin. The goal of the paper is to answer the question of finiteness for the above integral.

A simpler version of the above problem has a rich history in the literature, where it arises in connection with the growth rate of real analytic functions and decay rate of oscillatory integrals. In particular, for $g \equiv 1$ and in dimension $n = 2$, the problem of finiteness of the integral in (1.1) has been fully treated by Phong, Stein and Sturm [2], while the related problem of determining the oscillation index of a two-dimensional oscillatory integral dates back to Varchenko [5]. It would be of considerable interest, however, to tackle the issue in higher dimensions (i.e. $n \geqslant 3$), where the problem of convergence of the unweighted integral ($g \equiv 1$) is still poorly understood and finiteness results are, at best, partial (see [5]).

Weighted integrals of the form (1.1) sometimes arise from their unweighted higher dimensional analogues after a suitable change of coordinates, especially if the higher dimensional f comes equipped with certain symmetries that can be exploited to reduce the dimension. A case in point is the important counterexample to Arnold's problem given by Varchenko in the context of oscillatory integrals in $\mathbb{R}^3$ (see section 5 of [5]). In our situation, Varchenko's example translates to

$$\iiint_{B^3 \subset \mathbb{R}^3} \frac{dx_1\, dx_2\, dx_3}{\left|(\lambda x_1^2 + x_1^4 + x_2^2 + x_3^2)^2 + x_1^{4p} + x_2^{4p} + x_3^{4p}\right|^\delta}, \tag{1.2}$$

where λ is a real parameter and p is a sufficiently large natural number. Now, a few trivial size estimates coupled with a cylindrical change of coordinates transforms

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the above three-dimensional unweighted integral to a two-dimensional weighted one, given by

$$\iint_{B^2\subset\mathbb{R}^2} \frac{|y|\,dy\,dx}{|(y^2+x^4+\lambda x^2)^2+x^{4p}+y^{4p}|^{\delta}}. \tag{1.3}$$

Clearly, the integral in (1.2) converges if and only if the integral in (1.3) does. In general, the hope is that results for integrals of the form (1.1) would shed some light on the behavior of the higher-dimensional unweighted ones they arise from.

Another problem closely related to the unweighted integrals is the one of obtaining L^2 bounds for oscillatory integral operators with degenerate real-analytic phase. The case for $n = 2$ has been addressed by Phong and Stein in [1]. In section 3, the techniques used in their paper are adapted to settle the problem of local integrability of $|g|^{\epsilon}/|f|^{\delta}$.

The problem of finiteness of the integral in (1.1) also gives rise to several related problems, some of which in turn translate to results for unweighted integrals in higher dimensions. A few questions which arise naturally in this context are:

(a) When is the ratio $|g|^{\epsilon}/|f|^{\delta}$ an A_p weight in B?

(b) How does the integral in (1.1) behave under perturbations of f if g is held fixed?

We shall return to these questions in subsequent papers.

2. Notation, Definitions and Preliminary Observations

Let f and g be real-analytic functions in a neighborhood of the origin in $\mathbb{R}^2$ such that $f(0) = g(0) = 0$, $f, g \not\equiv 0$. Suppose the Taylor series expansions of f and g are given by

$$f(x,y) = \sum_{\substack{p,q=0\\(p,q)\neq(0,0)}}^{\infty} c_{pq}x^p y^q, \qquad g(x,y) = \sum_{\substack{p,q=0\\(p,q)\neq(0,0)}}^{\infty} \tilde{c}_{pq}x^p y^q.$$

The *Newton polyhedron* of $f(x,y)$ (respectively $g(x,y)$) at the origin is defined to be the convex hull of the union of all the north-east quadrants in $\mathbb{R}^2_{\geqslant 0}$ with corners at the points (p,q) satisfying $c_{pq} \neq 0$ (respectively $\tilde{c}_{pq} \neq 0$). The boundary of the Newton polyhedron is called the *Newton diagram*.

There is an alternative description of the Newton diagram which is sometimes more useful for analytical purposes. In view of the Weierstrass Preparation Theorem, f can be expressed, after a non-singular change of coordinates and up to a non-vanishing factor which we ignore, as a polynomial in y with coefficients analytic in x. Factoring out this polynomial, we write f as

$$f(x,y) = x^{\tilde{\alpha}_1} y^{\tilde{\beta}_1} \prod_{\nu\,;\,\nu\in I_f} (y - r_{\nu}(x)),$$

where $\tilde{\alpha}_1$ and $\tilde{\beta}_1$ are non-negative integers and the $r_{\nu}(x)$'s are the non-trivial zeros of the above-mentioned polynomial in y. Here, ν ranges over an index set I_f that is in one-to-one correspondence with the set of roots of f. In order to avoid confusion with subscripts later on, it is better to think of ν not as a positive integer but as the unique element of I_f associated to the root r_{ν}. In a small neighborhood of 0, the r_{ν}'s may be expressed as fractional power series, the so-called Puiseux series in

x, of the following form:

$$r_\nu(x) = c_\nu x^{a_\nu} + O\left(x^{b_\nu}\right),$$

where $b_\nu > a_\nu > 0$ are rational numbers and $c_\nu \neq 0$. In fact, each $r_\nu(x)$ is an analytic function in z, where $z = x^{1/M}$ for some positive integer M. A detailed discussion of the theory of existence and convergence of Puiseux series may be obtained in Siegel [4, pp. 90 - 98] and in Saks and Zygmund [3, pp. 268 - 271]. For a more concise treatment, see section 3 of [1].

We may also write, using similar arguments,

$$g(x,y) = x^{\tilde{\alpha}_2} y^{\tilde{\beta}_2} \prod_{\mu\,;\,\mu \in I_g} (y - s_\mu(x)),$$

where the s_μ's are the roots of g, and the leading exponent of s_μ is denoted by a_μ. We order the combined set of distinct exponents a_ν and a_μ into one increasing list of exponents a_l,

$$0 < a_1 < a_2 < \cdots < a_N,$$

and define

$$m_l := \#\{\nu : r_\nu(x) = c_\nu x^{a_l} + \cdots, \text{ for some } c_\nu \neq 0\}.$$

We call m_l the *generalized multiplicity* of f corresponding to the exponent a_l. The generalized multiplicity of g corresponding to a_l will be denoted by n_l. If a_l does not occur as a leading exponent of any root of f (respectively g), then we set $m_l = 0$ (respectively $n_l = 0$).

Now, it can be shown (for a proof see observation 1 in section 5 (b) of [1]) that the Newton diagram of f has vertices at the points (A_l, B_l), where

$$A_l = \tilde{\alpha}_1 + a_1 m_1 + a_2 m_2 + \cdots + a_l m_l, \qquad B_l = \tilde{\beta}_1 + m_{l+1} + \cdots + m_N,$$
$$C_l = \tilde{\alpha}_2 + a_1 n_1 + a_2 n_2 + \cdots + a_l n_l, \qquad D_l = \tilde{\beta}_2 + n_{l+1} + \cdots + n_N.$$

It follows that the leading exponents a_l of the roots of f (or g) can be read off from their respective Newton diagrams, together with their generalized multiplicities m_l. In fact, the boundary segment joining the vertices (A_{l-1}, B_{l-1}) and (A_l, B_l) has slope $\frac{-1}{a_l}$ and vertical drop m_l.

Next, we introduce the following notation. If $\mathcal{L}_l$ is a line of slope $\frac{-1}{a_l}$ that occurs in the Newton diagram of either f or g, then we denote the point of intersection of $\mathcal{L}_l$ with the bisectrix $p = q$ by

$$\begin{cases} \left(\delta_l{}^{-1}, \delta_l{}^{-1}\right) & \text{if } \mathcal{L}_l \text{ is a boundary line of the Newton diagram of } f, \\ \left(\tilde{\delta}_l^{-1}, \tilde{\delta}_l^{-1}\right) & \text{if } \mathcal{L}_l \text{ is a boundary line of the Newton diagram of } g. \end{cases}$$

The alternative description of the Newton diagram allows us to obtain an explicit formula for $\delta_l{}^{-1}$ and $\tilde{\delta}_l^{-1}$ in terms of a_l, A_l, B_l, C_l and D_l:

$$\delta_l = \frac{1 + a_l}{A_l + a_l B_l}, \qquad \tilde{\delta}_l = \frac{1 + a_l}{C_l + a_l D_l}. \tag{2.1}$$

The *Newton distance of* f, denoted by $\delta_0(f)$, is defined as follows:

$$\delta_0(f) := \min_l \delta_l.$$

The Newton distance plays an important role in problems involving the growth rate of the distribution function of f, local integrability of $|f|^{-\delta}$ and the decay rate of

L^2 norms of oscillatory integral operators with phase f. In particular, it is known that $|f|^{-\delta}$ is locally integrable in a neighborhood of the origin if $\delta < \delta_0(f)$ (after possibly a certain analytic change of coordinates), while the exact decay rate of an oscillatory integral operator on $\mathbb{R}$ with phase $f(x,y)$ and smooth non-degenerate cut-off $\chi(x,y)$ is given by $\delta_0(f_{xy})$. In this paper, we shall obtain an analogue of $\delta_0(f)$ in the weighted situation.

We conclude our preliminary discussion with the following observations:

(1) First, we note that the quantities $a_l, m_l, n_l, A_l, B_l, C_l, D_l, \delta_l, \tilde{\delta}_l$ are all coordinate dependent.

(2) Second, we observe that the above discussion allows us to extend the notion of Newton diagram to a certain class of functions strictly larger than the class of real-analytic functions. Let f(and g) be of the form $\prod_\kappa (y - t_\kappa(x))$ (where the t_κ's are convergent Puiseux series in x, though not necessarily the roots of an analytic function). Then the definitions of a_l, m_l, etc. still make sense, with the possibility that now the vertices (A_l, B_l) may be fractional, instead of integer points. Let us consider the set

$$\bigcup_l \{(x,y)\,;\, x \geqslant A_l,\, y \geqslant B_l\}$$

and call the boundary of the convex hull of the above union set the *generalized Newton diagram* of f. The geometric interpretation of $\left(\delta_l{}^{-1}, \delta_l{}^{-1}\right)$ and $\left(\tilde{\delta}_l^{-1}, \tilde{\delta}_l^{-1}\right)$ continues to hold, and we compute these objects using the same formulae as in (2.1).

(3) Finally, we claim that although $\delta_l{}^{-1}$(respectively $\tilde{\delta}_l^{-1}$) has been defined only for those values of l which are "represented" in the Newton diagram of f (respectively g), it is possible to define these quantities for all values of l, $1 \leqslant l \leqslant N$, using the convention that $m_l = 0$ (respectively $n_l = 0$) for any l that is "omitted" in f(respectively g). Geometrically, if $m_l = 0$, one imagines a line segment of length zero and of slope $\frac{-1}{a_l}$ in between the segments of slope $\frac{-1}{a_{l-1}}$ and $\frac{-1}{a_{l+1}}$ in the (generalized) Newton diagram of f, and defines $\left(\delta_l{}^{-1}, \delta_l{}^{-1}\right)$ to be the point of intersection of this "imaginary" line segment with the bisectrix $p = q$. The coordinates of the point of intersection are still obtained from the expressions in (2.1).

The statement of our main theorem requires the following definition:

Let us consider a coordinate transformation φ given by

$$(x,y) \mapsto (x, y - q(x)) \quad \text{or} \quad (x,y) \mapsto (x - q(y), y),$$

where q is a convergent *real-valued* Puiseux series in a neighborhood of the origin. We shall see in the proof of Theorem 1 that a change of coordinates of this type transforms a real-analytic function to a function of the form

$$\prod_\kappa (y - t_\kappa(x)),$$

to which the notion of a generalized Newton diagram may be applied. We shall call such a transformation φ "good" and denote the class of all good transformations by $\mathcal{C}$. For any $\varphi \in \mathcal{C}$, we define the *weighted Newton distance* of f and g associated to φ as follows:

$$\delta_0\,(g, f, \epsilon\,; \varphi) := \min_l \left[\delta_l(\varphi)\left(1 + \frac{\epsilon}{\tilde{\delta}_l(\varphi)}\right)\right],$$

where the index l runs through the combined set of slopes of boundary lines of the (generalized) Newton diagrams of f and g expressed in the coordinate system φ. $\delta_l(\varphi)$ and $\tilde{\delta}_l(\varphi)$ are the values of δ_l and $\tilde{\delta}_l$, respectively, computed in these coordinates.

3. Statement and Proof of Main Theorem

We are now ready to state and prove the main theorem.

Theorem 1. *Let f and g be real-analytic (complex-valued) functions in a neighborhood of the origin in $\mathbb{R}^2$ with $f(0) = g(0) = 0$. Then, there exists a small neighborhood V of the origin such that for every $\epsilon > 0$ the following holds:*

$$\int_V \frac{|g(x,y)|^\epsilon}{|f(x,y)|^\delta}\, dy\, dx < \infty$$

for any $\delta < \delta_0^{\mathcal{C}}(g, f, \epsilon)$, and

$$\int_V \frac{|g(x,y)|^\epsilon}{|f(x,y)|^\delta}\, dy\, dx = \infty$$

for any $\delta \geq \delta_0^{\mathcal{C}}(g, f, \epsilon)$, where

$$\delta_0^{\mathcal{C}}(g, f, \epsilon) := \min_{\varphi \in \mathcal{C}} \ \delta_0(g, f, \epsilon\,;\, \varphi).$$

In fact, there exists a finite subclass of $\mathcal{C}$, denoted by $\mathcal{C}_0$, which depends on f and g but is independent of ϵ, such that

$$\delta_0^{\mathcal{C}}(g, f, \epsilon) = \delta_0^{\mathcal{C}_0}(g, f, \epsilon) := \min_{\varphi \in \mathcal{C}_0} \ \delta_0(g, f, \epsilon\,;\, \varphi).$$

The class $\mathcal{C}_0$ contains, apart from the identity transformation, coordinate changes of the form $(x, y) \mapsto (x, y - q(x))$ and $(x, y) \mapsto (x - q(y), y)$ for certain Puiseux series q (depending on f or g) with leading exponents greater than or equal to 1.

Remark. We shall henceforth call any $\varphi \in \mathcal{C}_0$ an admissible change of coordinates. Our proof, in fact, provides an algorithm for computing all the admissible coordinate changes and for selecting the optimal one. One also concludes from the theorem that any $\varphi \in \mathcal{C}_0$, and in particular the "minimizing" transformation, can be at worst $C^{1+\epsilon_0}$ for some small $\epsilon_0 > 0$ depending on f and g.

Proof of Theorem 1. Let us for now assume that $a_1 \geqslant 1$. This results in no loss of generality in the proof that we are about to present, and we shall mention the routine modifications for the general case at the end. Also without loss of generality, we restrict ourselves to considering the integral only on the quadrant $\{x > 0, y > 0\}$, and inspect the integrand on the support of the cut-off functions $\chi_j(x)\chi_k(y)$, where $\chi_j(x)$ and $\chi_k(y)$ are the indicator functions of the sets

$$2^{-j-1} < x < 2^{-j+1} \quad \text{and} \quad 2^{-k-1} < y < 2^{-k+1}$$

respectively. Here j and k may be taken to be large. We need to consider the following ranges of j and k, depending on whether there is any cancellation between y and x^{a_l}:

$$k \leqslant a_1 j - K_1; \quad k \geqslant a_N j + K_N;$$
$$a_l j + K_l \leqslant k \leqslant a_{l+1} j - K_{l+1} \text{ for some } l \text{ with } 1 \leqslant l \leqslant N-1;$$
$$a_l j - K_l \leqslant k \leqslant a_l j + K_l \text{ for some } l \text{ with } 1 \leqslant l \leqslant N.$$

Here the K_l's are large constants depending only on the exponents a_l and possibly on the coefficients of the Puiseux series in the factorization of f and g. For convenience, we shall denote the first range by $k \ll a_1 j$, the second one by $k \gg a_N j$, the third category of ranges by $a_l j \ll k \ll a_{l+1} j$ and the fourth by $k \approx a_l j$. The binary relation $A \sim B$ is used in the following sense:

$$A \sim B \quad \text{iff} \quad \frac{1}{C_1} B \leq A \leq C_2 B$$

for large constants C_1 and C_2 depending on f and g alone.

Case 1: $k \ll a_1 j$. In this case,

$$|f(x,y)| \sim 2^{-k\left(\tilde{\beta}_1 + m_1 + m_2 + \cdots + m_N\right)} = 2^{-kB_0}, \qquad |g(x,y)| \sim 2^{-kD_0}.$$

Therefore,

$$\begin{aligned}
&\sum_{(k,j)\,;\,k \ll a_1 j} \int \frac{|g(x,y)|^{\epsilon}}{|f(x,y)|^{\delta}} \chi_j(x) \chi_k(y)\, dy\, dx \\
&\quad \sim \sum_{k \ll a_1 j} 2^{k(\delta B_0 - \epsilon D_0)} 2^{-k} 2^{-j} = \sum_{k \ll a_1 j} 2^{k(\delta B_0 - \epsilon D_0 - 1)} 2^{-j} \\
&\quad \sim \sum_{k} 2^{k(\delta B_0 - \epsilon D_0 - 1 - \frac{1}{a_1})},
\end{aligned}$$

which converges iff

$$\delta B_0 - \epsilon D_0 - 1 - \frac{1}{a_1} < 0,$$

i.e., iff

$$\begin{aligned}
\delta < \frac{\epsilon D_0 + 1 + \frac{1}{a_1}}{B_0} &= \frac{\epsilon a_1 D_0 + a_1 + 1}{a_1 B_0} \\
&= \frac{\epsilon \frac{a_1 D_0}{a_1 + 1} + 1}{\frac{a_1 B_0}{a_1 + 1}} = \delta_1 \left(1 + \frac{\epsilon}{\tilde{\delta}_1}\right).
\end{aligned}$$

The case $k \gg a_N j$ is treated in a similar way.

Case 2: $a_l j \ll k \ll a_{l+1} j$. In this case,

$$|y - r_\nu(x)| \sim \begin{cases} 2^{-a_\nu j} & \text{if } a_\nu \leqslant a_l, \\ 2^{-k} & \text{if } a_\nu > a_l. \end{cases}$$

We then have the following size estimates:

$$\begin{aligned}
|f(x,y)| &\sim 2^{-j(\tilde{\alpha}_1 + a_1 m_1 + \ldots + a_l m_l)} 2^{-k(\tilde{\beta}_1 + m_{l+1} + \ldots + m_N)} \\
&\sim 2^{-A_l j} 2^{-B_l k}.
\end{aligned}$$

Similarly,

$$|g(x,y)| \sim 2^{-C_l j} 2^{-D_l k}.$$

Therefore,

$$\begin{aligned}
&\sum_{a_l j \ll k \ll a_{l+1} j} \int \frac{|g(x,y)|^{\epsilon}}{|f(x,y)|^{\delta}} \chi_j(x) \chi_k(y)\, dy\, dx \\
&\quad \sim \sum_{a_l j \ll k \ll a_{l+1} j} 2^{j(\delta A_l - \epsilon C_l)} 2^{k(\delta B_l - \epsilon D_l)} 2^{-k} 2^{-j},
\end{aligned}$$

where the sum is over both indices j and k. To facilitate the summation, let us introduce the new dummy variable $\tilde{k}$ in place of k, defined as follows:

$$k = \tilde{k} + a_l j, \quad 0 \ll \tilde{k} \ll (a_{l+1} - a_l)j.$$

Then the above expression reduces to

$$\begin{aligned}\sum_{0\ll\tilde{k}\ll(a_{l+1}-a_l)j} & 2^{j(\delta A_l - \epsilon C_l - 1)} 2^{(\tilde{k}+a_l j)(\delta B_l - \epsilon D_l - 1)} \\ &= \sum_{0\ll\tilde{k}\ll(a_{l+1}-a_l)j} 2^{[\delta(A_l + a_l B_l) - \epsilon(C_l + a_l D_l) - a_l - 1]j} \times 2^{\tilde{k}(\delta B_l - \epsilon D_l - 1)},\end{aligned}$$

which converges iff

(a) $\delta(A_l + a_l B_l) - \epsilon(C_l + a_l D_l) - (a_l + 1) < 0$ when $\delta B_l - \epsilon D_l - 1 < 0$,

(b) $\delta(A_l + a_l B_l) - \epsilon(C_l + a_l D_l) - (a_l + 1) + (a_{l+1} - a_l)(\delta B_l - \epsilon D_l - 1) < 0$ when $\delta B_l - \epsilon D_l - 1 \geqslant 0$;

i.e. iff

$$\text{(a) } \delta < \frac{\epsilon\,(C_l + a_l D_l) + a_l + 1}{A_l + a_l B_l} = \delta_l\left(1 + \frac{\epsilon}{\tilde{\delta}_l}\right), \quad \text{when } \delta < \frac{\epsilon D_l + 1}{B_l},$$

$$\begin{aligned}\text{(b) } \delta < \frac{\epsilon\,(C_l + a_{l+1} D_l) + a_{l+1} + 1}{A_l + a_{l+1} B_l} &= \frac{\epsilon\,(C_{l+1} + a_{l+1} D_{l+1}) + a_{l+1} + 1}{A_{l+1} + a_{l+1} B_{l+1}} \\ &= \delta_{l+1}\left(1 + \frac{\epsilon}{\tilde{\delta}_{l+1}}\right), \quad \text{when } \delta \geqslant \frac{\epsilon D_l + 1}{B_l}.\end{aligned}$$

Now, a straightforward computation shows that the following conditions are equivalent:

(i) $\dfrac{\epsilon D_l + 1}{B_l} < \dfrac{\epsilon\,(C_l + a_{l+1} D_l) + a_{l+1} + 1}{A_l + a_{l+1} B_l} = \delta_{l+1}\left(1 + \dfrac{\epsilon}{\tilde{\delta}_{l+1}}\right)$;

(ii) $\dfrac{\epsilon D_l + 1}{B_l} < \dfrac{\epsilon\,(C_l + a_l D_l) + a_l + 1}{A_l + a_l B_l} = \delta_l\left(1 + \dfrac{\epsilon}{\tilde{\delta}_l}\right)$;

(iii) $\dfrac{\epsilon\,(C_l + a_{l+1} D_l) + a_{l+1} + 1}{A_l + a_{l+1} B_l} < \dfrac{\epsilon\,(C_l + a_l D_l) + a_l + 1}{A_l + a_l B_l}$;

(iv) $\dfrac{\epsilon\,(C_l + a_{l+1} D_l) + a_{l+1} + 1}{A_l + a_{l+1} B_l} < \dfrac{\epsilon C_l + 1}{A_l}$;

(v) $\dfrac{\epsilon\,(C_l + a_l D_l) + a_l + 1}{A_l + a_l B_l} < \dfrac{\epsilon C_l + 1}{A_l}$;

(vi) $\dfrac{\epsilon D_l + 1}{B_l} < \dfrac{\epsilon C_l + 1}{A_l}$.

So, there are two possibilities:

(a) $$\frac{\epsilon D_l + 1}{B_l} < \frac{\epsilon C_l + 1}{A_l},$$

in which case the necessary and sufficient conditions in (i) - (vi) lead to the string of inequalities

$$\frac{\epsilon D_l + 1}{B_l} < \delta_{l+1}\left(1 + \frac{\epsilon}{\tilde{\delta}_{l+1}}\right) < \delta_l\left(1 + \frac{\epsilon}{\tilde{\delta}_l}\right) < \frac{\epsilon C_l + 1}{A_l}.$$

Therefore the optimal δ satisfies

$$\delta < \delta_{l+1}\left(1+\frac{\epsilon}{\tilde{\delta}_{l+1}}\right).$$

The second possibility is

$$\text{(b)} \qquad \frac{\epsilon D_l+1}{B_l} \geqslant \frac{\epsilon C_l+1}{A_l},$$

in which case

$$\frac{\epsilon D_l+1}{B_l} \geqslant \delta_{l+1}\left(1+\frac{\epsilon}{\tilde{\delta}_{l+1}}\right) \geqslant \delta_l\left(1+\frac{\epsilon}{\tilde{\delta}_l}\right) \geqslant \frac{\epsilon C_l+1}{A_l},$$

and the optimal δ satisfies

$$\delta < \delta_l\left(1+\frac{\epsilon}{\tilde{\delta}_l}\right).$$

In either case, we have

$$\delta < \min_l \left[\delta_l\left(1+\frac{\epsilon}{\tilde{\delta}_l}\right)\right],$$

where δ_l and $\tilde{\delta}_l$ are computed based on f and g in the original set of coordinates, i.e. under the identity transformation $(x,y)\mapsto(x,y)$.

Case 3: $k \approx a_l j$. This range of j and k depicts the region "near" the cluster of roots of f and g that has a common leading exponent a_l. For the sake of a more organized notation to be made clear shortly, let us rename $l = l_1$. We now fix a *real* root q_1 of the above cluster, such that

$$q_1(x) = c_{l_1}^{\alpha_1} x^{a_{l_1}} + \text{ higher order terms},$$

for some index α_1 with $c_{l_1}^{\alpha_1} \neq 0$ (the case of complex-valued roots will be treated shortly). Since q_1 is real-valued for all x in its domain, $c_{l_1}^{\alpha_1} \in \mathbb{R}$. Here, q_1 could be the Puiseux series of a root of f and/or g. For a real-analytic function h, which in our case might be either f or g, we introduce the following notation:

$$\begin{aligned}
\mathcal{S}_{l_1}(h) &:= \big\{q\,;q \text{ is the Puiseux series corresponding to a root of } h \\
&\qquad\qquad\qquad\qquad \text{with leading exponent } a_{l_1}\big\},\\
\mathcal{S}_{l_1}^{\alpha_1}(h) &:= \big\{q \in \mathcal{S}_{l_1}(h)\,;\, q \text{ has leading term } c_{l_1}^{\alpha_1}x^{a_{l_1}}\big\},\\
m_{l_1}(h) &:= \text{ number of roots in } \mathcal{S}_{l_1}(h)\text{, counted with their multiplicities},\\
m_{l_1}^{\alpha_1}(h) &:= \text{ number of roots in } \mathcal{S}_{l_1}^{\alpha_1}(h)\text{, counted with their multiplicities}.
\end{aligned}$$

Then,

$$\mathcal{S}_{l_1}(h) = \bigcup_{\alpha_1} \mathcal{S}_{l_1}^{\alpha_1}(h) \quad \text{and} \quad m_{l_1}(h) = \sum_{\alpha_1} m_{l_1}^{\alpha_1}(h).$$

For simplicity, we shall denote

$$m_{l_1}^{\alpha_1}(f) = m_{l_1}^{\alpha_1}, \quad m_{l_1}^{\alpha_1}(g) = n_{l_1}^{\alpha_1}, \quad \mathcal{S}_{l_1}(f)\cup\mathcal{S}_{l_1}(g) = \mathcal{S}_{l_1}.$$

Thus, for every $q \in \mathcal{S}_{l_1}^{\alpha_1}$ such that $q \not\equiv q_1$, there exist $a_{l_1 l_2}^{\alpha_1} > a_{l_1}$ and $c_{l_1 l_2}^{\alpha_1\alpha_2} \neq 0$ such that

$$q(x) - q_1(x) = c_{l_1 l_2}^{\alpha_1\alpha_2} x^{a_{l_1 l_2}^{\alpha_1}} + \text{ higher order terms}.$$

We assume that the $a^{\alpha_1}_{l_1 l_2}$'s have been arranged in increasing order of magnitude, i.e.

$$a_{l_1} = a_{l_1 0} < a^{\alpha_1}_{l_1 1} < a^{\alpha_1}_{l_1 2} < \cdots < a^{\alpha_1}_{l_1 l_2} < a^{\alpha_1}_{l_1 (l_2+1)} < \cdots < \infty\,. \tag{3.1}$$

Define

$$\mathcal{S}^{\alpha_1}_{l_1 l_2}(h; q_1) := \{q \in \mathcal{S}^{\alpha_1}_{l_1}(h; q_1);\ (q - q_1) \text{ has leading exponent } a^{\alpha_1}_{l_1 l_2}\}$$

and

$m^{\alpha_1}_{l_1 l_2}(h; q_1) :=$ number of roots in $\mathcal{S}^{\alpha_1}_{l_1 l_2}(h; q_1)$, counted with their multiplicities.

As before, set

$$m^{\alpha_1}_{l_1 l_2}(f) = m^{\alpha_1}_{l_1 l_2}, \quad m^{\alpha_1}_{l_1 l_2}(g) = n^{\alpha_1}_{l_1 l_2}, \quad \mathcal{S}^{\alpha_1}_{l_1 l_2}(f; q_1) \cup \mathcal{S}^{\alpha_1}_{l_1 l_2}(g; q_1) = \mathcal{S}^{\alpha_1}_{l_1 l_2}.$$

Then,

$$\sum_{l_2 \geqslant 1} m^{\alpha_1}_{l_1 l_2}(h; q_1) = m^{\alpha_1}_{l_1}(h; q_1) - M_1(h),$$

where $M_1(h)$ is the multiplicity of q_1 as a root of h. In order to examine the integral in a neighborhood of q_1, we set

$$|y - q_1(x)| \sim 2^{-p}, \quad p \geqslant a_{l_1} j.$$

Let us observe that this is equivalent to making the change of variables $x \mapsto x$ and $y \mapsto y - q_1(x)$. We denote this change of variables by φ. φ will be a member of the class of admissible coordinates $\mathcal{C}_0$. Note that φ preserves the origin and has Jacobian identically equal to 1. In the subsequent analysis, the role of k, as depicted in cases 1 and 3, is taken over by p. For example, we shall now need to consider the following ranges of p and j:

$$a^{\alpha_1}_{l_1 l_2} j \ll p \ll a^{\alpha_1}_{l_1 (l_2+1)} j, \qquad p \approx a^{\alpha_1}_{l_1 l_2} j.$$

Subcase 1: $a^{\alpha_1}_{l_1 l_2} j \ll p \ll a^{\alpha_1}_{l_1 (l_2+1)} j$. In this region, the factors of f and g satisfy the following estimates:

$$|y - q(x)| = |(y - q_1(x)) - (q(x) - q_1(x))| \sim \begin{cases} 2^{-a_{l_1'} j}, & \text{if } q \in \mathcal{S}_{l_1'}, l_1' < l_1\ ; \\ 2^{-a_{l_1} j}, & \text{if } q \in \mathcal{S}_{l_1'}, l_1' > l_1\ ; \\ 2^{-a_{l_1} j}, & \text{if } q \in \mathcal{S}_{l_1} \setminus \mathcal{S}^{\alpha_1}_{l_1}; \\ 2^{-a^{\alpha_1}_{l_1 l_2'} j}, & \text{if } q \in \mathcal{S}^{\alpha_1}_{l_1 l_2'}, l_2' \leqslant l_2; \\ 2^{-p}, & \text{if } q \in \mathcal{S}^{\alpha_1}_{l_1 l_2'}, l_2' > l_2; \\ 2^{-p}, & \text{if } q = q_1. \end{cases}$$

Therefore,

$$|f(x, y)| \sim 2^{-(A_{l_1-1} + a_{l_1} B_{l_1}) j} 2^{-a_{l_1}(m_{l_1} - m^{\alpha_1}_{l_1}) j} 2^{-A^{\alpha_1}_{l_1 l_2} j} 2^{-p B^{\alpha_1}_{l_1 l_2}},$$

where

$$A^{\alpha_1}_{l_1 l_2} = \sum_{l_2' \leqslant l_2} a^{\alpha_1}_{l_1 l_2'} m^{\alpha_1}_{l_1 l_2'}, \qquad B^{\alpha_1}_{l_1 l_2} = \sum_{l_2' > l_2} m^{\alpha_1}_{l_1 l_2'}\ ;$$

i.e.

$$|f(x, y)| \sim 2^{-(A_{l_1-1} + a_{l_1} B_{l_1-1} - a_{l_1} m^{\alpha_1}_{l_1} + A^{\alpha_1}_{l_1 l_2}) j} 2^{-B^{\alpha_1}_{l_1 l_2} p},$$

$$|g(x, y)| \sim 2^{-(C_{l_1-1} + a_{l_1} D_{l_1-1} - a_{l_1} n^{\alpha_1}_{l_1} + C^{\alpha_1}_{l_1 l_2}) j} 2^{-D^{\alpha_1}_{l_1 l_2} p}.$$

This implies that,

$$\sum_{\substack{(j,k,p)\,;\,k\sim a_{l_1}j\\ a^{\alpha_1}_{l_1l_2}j\ll p\ll a^{\alpha_1}_{l_1(l_2+1)}j}} \int \frac{|g(x,y)|^{\epsilon}}{|f(x,y)|^{\delta}}\chi_j(x)\chi_k(y)\chi_p(y-q_1(x))\,dy\,dx$$

$$\sim \sum_{\substack{(j,k,p)\,;\,k\sim a_{l_1}j\\ a^{\alpha_1}_{l_1l_2}j\ll p\ll a^{\alpha_1}_{l_1(l_2+1)}j}} \frac{2^{-(C_{l_1-1}+a_{l_1}D_{l_1-1}-a_{l_1}n^{\alpha_1}_{l_1}+C^{\alpha_1}_{l_1l_2})j\epsilon}2^{-D^{\alpha_1}_{l_1l_2}p\epsilon}}{2^{-(A_{l_1-1}+a_{l_1}B_{l_1-1}-a_{l_1}m^{\alpha_1}_{l_1}+A^{\alpha_1}_{l_1l_2})j\delta}2^{-B^{\alpha_1}_{l_1l_2}p\delta}}\,2^{-p}2^{-j}$$

$$\sim \sum_{\substack{(j,p)\,;\\ a^{\alpha_1}_{l_1l_2}j\ll p\ll a^{\alpha_1}_{l_1(l_2+1)}j}} 2^{j\delta(A_{l_1-1}+a_{l_1}B_{l_1-1}-a_{l_1}m^{\alpha_1}_{l_1}+A^{\alpha_1}_{l_1l_2})}$$
$$\times\, 2^{-j\left[\epsilon(C_{l_1-1}+a_{l_1}D_{l_1-1}-a_{l_1}n^{\alpha_1}_{l_1}+C^{\alpha_1}_{l_1l_2})+1\right]}2^{p(B^{\alpha_1}_{l_1l_2}\delta-D^{\alpha_1}_{l_1l_2}-1)}.$$

Just as before, we can show that the infinite series converges:

(a) for $\delta B^{\alpha_1}_{l_1l_2}-\epsilon D^{\alpha_1}_{l_1l_2}-1\leqslant 0$ iff

$$\delta<\frac{\epsilon\left(C_{l_1-1}+a_{l_1}D_{l_1-1}-a_{l_1}n^{\alpha_1}_{l_1}+C^{\alpha_1}_{l_1l_2}+a^{\alpha_1}_{l_1l_2}D^{\alpha_1}_{l_1l_2}\right)+a^{\alpha_1}_{l_1l_2}+1}{A_{l_1-1}+a_{l_1}B_{l_1-1}-a_{l_1}m^{\alpha_1}_{l_1}+A^{\alpha_1}_{l_1l_2}+a^{\alpha_1}_{l_1l_2}B^{\alpha_1}_{l_1l_2}} :=\omega(q_1;l_1,l_2;\alpha_1);$$

(b) for $\delta B^{\alpha_1}_{l_1l_2}-\epsilon D^{\alpha_1}_{l_1l_2}-1>0$ iff

$$\delta<\frac{\epsilon\left(C_{l_1-1}+a_{l_1}D_{l_1-1}-a_{l_1}n^{\alpha_1}_{l_1}+C^{\alpha_1}_{l_1l_2}+a^{\alpha_1}_{l_1l_2}D^{\alpha_1}_{l_1(l_2+1)}\right)+a^{\alpha_1}_{l_1(l_2+1)}+1}{A_{l_1-1}+a_{l_1}B_{l_1-1}-a_{l_1}m^{\alpha_1}_{l_1}+A^{\alpha_1}_{l_1(l_2+1)}+a^{\alpha_1}_{l_1(l_2+1)}B^{\alpha_1}_{l_1(l_2+1)}} :=\omega(q_1;l_1,(l_2+1);\alpha_1).$$

We observe that

$$\left(\frac{a^{\alpha_1}_{l_1l_2}+1}{A_{l_1-1}+a_{l_1}B_{l_1-1}-a_{l_1}m^{\alpha_1}_{l_1}+A^{\alpha_1}_{l_1l_2}+a^{\alpha_1}_{l_1l_2}B^{\alpha_1}_{l_1l_2}}\right)^{-1} := \left(\delta^{\alpha_1}_{l_1l_2}\right)^{-1}$$

is the point of intersection of the bisectrix $p=q$ with the face of the generalized Newton diagram of the function $\tilde f(x,y)=f\circ\varphi(x,y)=f(x,y-q_1(x))$ that has slope $\frac{-1}{a^{\alpha_1}_{l_1l_2}}$. Denoting by $(\tilde\delta^{\alpha_1}_{l_1l_2})^{-1}$ the corresponding quantity for g, we get

$$\omega(q_1;l_1,l_2;\alpha_1)=\delta^{\alpha_1}_{l_1l_2}\left(1+\frac{\epsilon}{\tilde\delta^{\alpha_1}_{l_1l_2}}\right).$$

Now, using the same techniques as in case 2, we see that the optimal exponent δ satisfies

$$\delta<\min_{l}\,\delta_l(\varphi)\left(1+\frac{\epsilon}{\tilde\delta_l(\varphi)}\right).$$

Subcase 2: $p\gg a^{\alpha_1}_{l_1L_2}j$, where $L_2=\max l_2$. This is the step that deals with the region "close" to the root q_1 and highlights the cancellation (if any) of the factor

$(y - q_1(x))$ between $|g|^\epsilon$ and $|f|^\delta$. In this region, f and g satisfy the following size estimates:

$$|f(x,y)| \sim 2^{-(A_{l_1-1}+a_{l_1}B_{l_1})j} 2^{-a_{l_1}(m_{l_1}-m_{l_1}^{\alpha_1})j} 2^{-A^{\alpha_1}_{l_1 L_2} j} 2^{-pM_1(f)},$$

$$|g(x,y)| \sim 2^{-(C_{l_1-1}+a_{l_1}D_{l_1})j} 2^{-a_{l_1}(n_{l_1}-n_{l_1}^{\alpha_1})j} 2^{-C^{\alpha_1}_{l_1 L_2} j} 2^{-pM_1(g)}.$$

So routine computations yield

$$\sum_{\substack{(j,k,p)\,;\,k\sim a_l j\\ p \gg a^{\alpha_1}_{l_1 L_2} j}} \int \frac{|g(x,y)|^\epsilon}{|f(x,y)|^\delta} \chi_j(x)\chi_k(y)\chi_p(y - q_1(x))\,dy\,dx$$

$$\sim \sum_{\substack{(j,k,p)\,;\,k\sim a_l j\\ p \gg a^{\alpha_1}_{l_1 L_2} j}} 2^{p(\delta M_1(f) - \epsilon M_1(g) - 1)} 2^{j\delta(A_{l_1-1}+a_{l_1}B_{l_1-1}-a_{l_1}m^{\alpha_1}_{l_1}+A^{\alpha_1}_{l_1 L_2})} \times 2^{-j\left[\epsilon(C_{l_1-1}+a_{l_1}D_{l_1-1}-a_{l_1}n^{\alpha_1}_{l_1}+C^{\alpha_1}_{l_1 L_2})+1\right]},$$

which converges iff

$$\delta < \min\left(\frac{\epsilon M_1(g)+1}{M_1(f)}, \omega(q_1; l_1, L_2; \alpha_1)\right).$$

We observe that $(\epsilon M_1(g)+1)/M_1(f)$ is the value of ω corresponding to the horizontal faces of the generalized Newton diagrams of $\tilde f$ and $\tilde g$.

Subcase 3: $p \approx a^{\alpha_1}_{l_1 l_2} j$. This range of indices depicts the region near the cluster of roots $\mathcal{S}^{\alpha_1}_{l_1 l_2}$. Let us fix a *real* root q_2 in this cluster. Then,

$$q_2(x) - q_1(x) = c^{\alpha_1\alpha_2}_{l_1 l_2} x^{a^{\alpha_1}_{l_1 l_2}} + \text{ higher order terms}$$

for some index α_2 such that $c^{\alpha_1\alpha_2}_{l_1 l_2} \neq 0$.

Defining

$$\mathcal{S}^{\alpha_1\alpha_2}_{l_1 l_2}(h; q_1) := \{q \in \mathcal{S}^{\alpha_1}_{l_1 l_2}(h; q_1);\ (q - q_1)(x) = c^{\alpha_1\alpha_2}_{l_1 l_2} x^{a^{\alpha_1}_{l_1 l_2}} + \text{ higher order terms}\},$$

and setting

$$\mathcal{S}^{\alpha_1\alpha_2}_{l_1 l_2} := \mathcal{S}^{\alpha_1\alpha_2}_{l_1 l_2}(f; q_1) \cup \mathcal{S}^{\alpha_1\alpha_2}_{l_1 l_2}(g; q_1),$$

we see that this subcase is identical to case 3, except that we now view the original cluster of roots under a finer resolution; namely, we consider only those roots which have the same leading term as q_1 and whose difference from q_1 has a leading factor of $c^{\alpha_1\alpha_2}_{l_1 l_2} x^{a^{\alpha_1}_{l_1 l_2}}$.

Thus, as before, we have that for any $q \in \mathcal{S}^{\alpha_1\alpha_2}_{l_1 l_2}$ such that $q \not\equiv q_2$, there exist $a^{\alpha_1\alpha_2}_{l_1 l_2 l_3} > a^{\alpha_1}_{l_1 l_2}$ and $c^{\alpha_1\alpha_2\alpha_3}_{l_1 l_2 l_3} \neq 0$ such that

$$(q - q_2)(x) = c^{\alpha_1\alpha_2\alpha_3}_{l_1 l_2 l_3} x^{a^{\alpha_1\alpha_2}_{l_1 l_2 l_3}} + \text{ higher order terms.}$$

Let us order

$$a^{\alpha_1}_{l_1 l_2} < a^{\alpha_1\alpha_2}_{l_1 l_2 1} < \cdots < a^{\alpha_1\alpha_2}_{l_1 l_2 l_3} < a^{\alpha_1\alpha_2}_{l_1 l_2 (l_3+1)} < \cdots < \infty.$$

Now, setting $y - q_2(x) \sim 2^{-p_2}$, $p_2 \geqslant p$ (in other words, making the transformation $x \mapsto x$, $y \mapsto y - q_2(x)$), we may treat this subcase exactly as the parent case 3 by further subdividing into cases depending on the range of p_2, i.e.

$$a^{\alpha_1\alpha_2}_{l_1 l_2 l_3} j \ll p_2 \ll a^{\alpha_1\alpha_2}_{l_1 l_2 (l_3+1)} j \quad \text{and} \quad p_2 \sim a^{\alpha_1\alpha_2}_{l_1 l_2 l_3} j.$$

The second type of region gives rise to further subcases of its own (identical to the ones already discussed but with an added degree of refinement in the cluster). In general, one considers, following an inductive scheme, progressively finer clusters given by

$$\begin{aligned}(3.2)\quad &\mathcal{S}^{\alpha_1\alpha_2\ldots\alpha_{s-1}}_{l_1l_2\ldots l_s}\\ &\quad := \mathcal{S}^{\alpha_1\alpha_2\ldots\alpha_{s-1}}_{l_1l_2\ldots l_s}(f;q_1,q_2,\cdots,q_{s-1}) \cup \mathcal{S}^{\alpha_1\alpha_2\ldots\alpha_{s-1}}_{l_1l_2\ldots l_s}(g;q_1,q_2,\cdots,q_{s-1})\end{aligned}$$

and

$$(3.3)\quad \mathcal{S}^{\alpha_1\alpha_2\ldots\alpha_s}_{l_1l_2\ldots l_s} := \mathcal{S}^{\alpha_1\alpha_2\ldots\alpha_s}_{l_1l_2\ldots l_s}(f;q_1,q_2,\cdots,q_{s-1}) \cup \mathcal{S}^{\alpha_1\alpha_2\ldots\alpha_s}_{l_1l_2\ldots l_s}(g;q_1,q_2,\cdots,q_{s-1}),$$

where the sets in (3.2) and (3.3) are defined recursively as follows.

Given $\mathcal{S}^{\alpha_1\cdots\alpha_{s-1}}_{l_1\cdots l_{s-1}}(h;q_1,\cdots,q_{s-2})$, we set

$$\begin{aligned}\mathcal{S}^{\alpha_1\alpha_2\cdots\alpha_{s-1}}_{l_1l_2\cdots l_s}(h;q_1,q_2,\ldots,q_{s-1}) := \big\{q \in \mathcal{S}^{\alpha_1\alpha_2\cdots\alpha_{s-1}}_{l_1l_2\cdots l_{s-1}}(h;q_1,q_2,\cdots,q_{s-2});\\ (q-q_{s-1})(x) \text{ has leading exponent } a^{\alpha_1\alpha_2\cdots\alpha_{s-1}}_{l_1l_2\cdots l_s}\big\}\end{aligned}$$

and

$$\begin{aligned}\mathcal{S}^{\alpha_1\alpha_2\cdots\alpha_s}_{l_1l_2\cdots l_s}(h;q_1,\cdots,q_{s-1}) := \{q \in \mathcal{S}^{\alpha_1\alpha_2\cdots\alpha_{s-1}}_{l_1l_2\cdots l_s}(h;q_1,\cdots,q_{s-1});\\ (q-q_{s-1})(x) = c^{\alpha_1\alpha_2\cdots\alpha_s}_{l_1l_2\cdots l_s}x^{a^{\alpha_1\alpha_2\cdots\alpha_{s-1}}_{l_1l_2\cdots l_s}} + \text{higher order terms, } c^{\alpha_1\alpha_2\cdots\alpha_s}_{l_1l_2\cdots l_s} \neq 0\}.\end{aligned}$$

Here $h = f$ or g, and q_i is a fixed real root of the cluster $\mathcal{S}^{\alpha_1\alpha_2\cdots\alpha_i}_{l_1l_2\cdots l_i}$, for $1 \leqslant i \leqslant s-1$. One observes that

$$\mathcal{S}_{l_1} \supseteq \mathcal{S}^{\alpha_1}_{l_1} \supsetneq \mathcal{S}^{\alpha_1}_{l_1l_2} \supseteq \mathcal{S}^{\alpha_1\alpha_2}_{l_1l_2} \supsetneq \cdots,$$

so the process terminates in a finite number of steps, when each of the final clusters $\mathcal{S}^{\alpha_1\alpha_2\cdots\alpha_s}_{l_1l_2\cdots l_s}$ consists of roots that are identical, i.e. when the degree of resolution is high enough to distinguish between any two distinct roots.

Case 4: Complex roots. We shall now indicate how the above arguments are modified in the presence of complex roots. Let a factor of f or g be given by $y - q_0(x)$, and let

$$q_0(x) = cx^{a_l} + \text{higher powers}.$$

Here $l = l_1$ and $c = c^{\alpha_1}_{l_1}$ for some α_1. There are two possibilities: $c \in \mathbb{R}$ and $c \notin \mathbb{R}$.

For $c \in \mathbb{R}$,

$$\begin{aligned}|y - q_0(x)| &\sim |y - \Re q_0(x)| + |\Im q_0(x)|\\ &\sim |y - \Re q_0(x)| + |x|^{b_{l_1}} \quad \text{for some } b_{l_1} > a_{l_1}.\end{aligned}$$

Now, in the ranges $k \ll a_{l_1}j$ and $k \gg a_{l_1}j$, the bounds for f and g from above and below are unaffected by the presence of $|x|^{b_{l_1}}$, since there is no cancellation between y and $|x|^{a_{l_1}}$, and $|x|^{a_{l_1}}$ is much bigger than $|x|^{b_{l_1}}$. Thus we need only consider the range $k \sim a_{l_1}j$. We set

$$y - \Re q_0(x) \sim 2^{-m}, \quad m \geqslant a_{l_1}j.$$

Then,

$$|y - q_0(x)| \sim 2^{-\min(b_{l_1}j,m)}.$$

Let us consider the collection of leading exponents of $q(x) - \Re q_0(x)$, where $q \in \mathcal{S}_{l_1}^{\alpha_1}$, and order them as in (3.1). We know that b_{l_1} occurs in this sequence, as it is the leading exponent of $q_0(x) - \Re q_0(x)$, so let us assume

$$b_{l_1} = a_{l_1 l_2}^{\alpha_1} \quad \text{for some } l_2.$$

Now, depending on the relative sizes of m and $b_{l_1} j$, we consider subcases of the following forms:

$$\left\{ \begin{array}{ll} \left. \begin{cases} (1)\ b_{l_1} j = a_{l_1 l_2}^{\alpha_1} j \leqslant a_{l_1 l_2'}^{\alpha_1} j \ll m \ll a_{l_1 (l_2'+1)}^{\alpha_1} j & \text{for } l_2' \geqslant l_2 \\ (2) \qquad\qquad m \approx a_{l_1 l_2'}^{\alpha_1} j & \text{for } l_2' \geqslant l_2 \end{cases} \right\} & \text{if } \ m \geqslant b_{l_1} j; \\ \left. \begin{cases} (3)\ a_{l_1 l_2'}^{\alpha_1} j \ll m \ll a_{l_1 (l_2'+1)}^{\alpha_1} \leq a_{l_1 l_2}^{\alpha_1} j = b_{l_1} j & \text{for } l_2' < l_2 \\ (4) \qquad\qquad m \approx a_{l_1 l_2'}^{\alpha_1} j & \text{for } l_2' \leqslant l_2 \end{cases} \right\} & \text{if } \ m \leqslant b_{l_1} j. \end{array} \right.$$

It is a matter of direct calculation to verify that the size estimates of f and g that one obtains in cases of the form (1) and (3) are identical to the ones obtained in subcase 1 of case 3, while the treatment of cases (2) and (4) exactly matches that of subcase 3 of case 3. Thus, a repetition of the same computations show that the optimal δ satisfies the following inequality:

$$\delta < \min_{l} \left[\delta_l(\tilde{\varphi}) \left(1 + \frac{\epsilon}{\tilde{\delta}_l(\tilde{\varphi})} \right) \right],$$

where $\tilde{\varphi}$ is an admissible change of coordinates given by $(x, y) \mapsto (x, y - \Re q_0(x))$.

This concludes our treatment of the case when $c \in \mathbb{R}$. The other case is in fact simpler than the preceding one, because now we have

$$|y - q_0(x)| \sim (|y| + |x|^{a_{l_1}})$$

The size estimates for f and g are, of course, unaffected in the range $a_{l_1} j \ll k \ll a_{l_1+1} j$. Even in the range $k \sim a_{l_1} j$, there can be no cancellation between the terms $|y|$ and $|x|^{a_{l_1}}$, so this range does not require a separate treatment. The case of complex roots has therefore been verified.

It is now possible to list all the members of $\mathcal{C}_0$, as follows:

$$\mathcal{C}_0 := \left\{ \begin{array}{l} \varphi : (x, y) \mapsto (x', y')\,;\ \varphi \text{ may be of the following forms:} \\ \text{either } \begin{cases} x' &= x \\ y' &= y - q(x) \end{cases} \quad \text{or} \quad \begin{cases} x' &= x - q(y) \\ y' &= y \end{cases} \\ \text{where } q \text{ is a real-valued Puiseux series with leading exponent} \geqslant 1. \\ y - q(x) \text{ or } x - q(y) \text{ may be either a real root of } f \text{ or } g \text{, or the} \\ \text{real part of a complex root of } f \text{ or } g \text{, provided the complex root} \\ \text{has a real leading coefficient.} \end{array} \right\}.$$

The proof shows that the δ given by

$$\delta_0^{\mathcal{C}_0}(g, f, \epsilon) := \min_{\varphi \in \mathcal{C}_0} \delta_0(g, f, \epsilon; \varphi)$$

is, in fact, optimal. Moreover, it is not hard to see that the above analysis may be replicated even if we set

$$y - t(x) \sim 2^{-p},$$

where t is a convergent Puiseux series, not necessarily associated to f or g. This shows that, in fact,

$$\delta_0^{\mathcal{C}}(g,f,\epsilon) = \delta_0^{\mathcal{C}_0}(g,f,\epsilon).$$

To complete the proof of Theorem 1, we need to justify the assumption $a_l \geqslant 1$. This is clearly the case in the range $k \geqslant j-K$ for a large enough constant K. In the remaining range $k \leqslant j-K$ it suffices to write f and g as polynomials in x with analytic coefficients in y (up to the usual non-vanishing factors). The zeroes of f and g are then of the form $x = \tilde{r}(y)$, where $\tilde{r}(y)$ is a Puiseux series in y with leading exponent greater than or equal to 1. We can now repeat our arguments with the roles of x and y interchanged. The proof of Theorem 1 is therefore complete. □

Remark. It is interesting to note the contrast of the weighted situation with the unweighted case, where the transformation yielding the optimal δ always turned out to be analytic [2]. For the weighted integral, the change of coordinates required to achieve the optimal δ, while being algebraic and indeed of a very special form, need not be better than once continuously differentiable, as the following example shows:

Let us consider the integral

$$I = \iint_B \frac{\left|(y^3-x^4)(y-x^2)^3\right|}{\left|(y^3-x^4)(y^3-x^4-x^5)(y-x^2)\right|^{\delta}}\, dy\, dx.$$

It is easy to see that the optimal δ is 1 and is obtained using the transformation $(x,y) \mapsto (x, y - x^{\frac{4}{3}}(1+x)^{\frac{1}{3}})$. Moreover, no analytic transformation of the form $(x,y) \mapsto (x, y - s(x))$ works, as can be seen by setting $s(x) = c_1 x^p + c_2 x^q + \cdots$, where $q > p \geqslant 1$ are integers, and computing directly the quantity $\min_l \delta_l \left(1 + \tilde{\delta}_l^{-1}\right)$. This involves having to consider various ranges of values of c_1, c_2 and p, whence one obtains that

$$\min_l \delta_l \left(1 + \tilde{\delta}_l^{-1}\right) = \left\{\begin{array}{ll} \dfrac{8}{7} & \text{if } p = 1; \\ \min\left(\dfrac{31}{28}, \dfrac{4q+5}{q+8}\right) & \text{if } p = 2, c_1 = 1, c_2 \neq 0; \\ \dfrac{31}{28} & \text{otherwise,} \end{array}\right\}$$

all of which are larger than 1.

Example. Finally, we would like to analyze Varchenko's counterexample, which was one of the motivating factors for the weighted integral, in this set-up. When λ is non-negative, the only admissible transformation for the integral in (1.3) is the identity transformation, as the denominator has neither a real root, nor a complex root with a real leading coefficient.

Thus, when $\lambda = 0$, the Newton diagram of f consists of a single boundary segment joining the points $(0,4)$ and $(8,0)$. One has

$$\delta_1 = \frac{3}{8}, \quad \tilde{\delta}_1 = \frac{3}{2}.$$

Therefore,

$$\delta_0^{\mathcal{C}}(g,f,1) = \frac{3}{8}\left(1 + \frac{2}{3}\right) = \frac{5}{8}.$$

When $\lambda > 0$, the Newton diagram of f again has a single boundary segment, joining $(0,4)$ and $(4,0)$, and one obtains

$$\delta_1 = \frac{1}{2}, \quad \tilde{\delta}_1 = 2.$$

This implies

$$\delta_0^{\mathcal{C}}(g,f,1) = \frac{1}{2}\left(1+\frac{1}{2}\right) = \frac{3}{4}.$$

When $\lambda < 0$, there exist non-trivial admissible transformations, all of which are of the form $(x,y) \mapsto (x, y-q(x))$, for some Puiseux series q with leading exponent 1. For any non-trivial $\varphi \in \mathcal{C}_0$, the generalized Newton diagram of $f \circ \varphi$ therefore has two boundary segments–one joining (0,4) and (2,2), and the other joining (2,2) and ($4p$,0). The generalized Newton diagram of $g \circ \varphi$ has a single segment, joining (0,1) and (1,0). The computations yield

$$\delta_1(\varphi) = \frac{1}{2}, \quad \tilde{\delta}_1(\varphi) = 2, \quad \delta_2(\varphi) = \frac{1}{2}, \quad \tilde{\delta}_2(\varphi) = 2p.$$

Therefore,

$$\delta_0^{\mathcal{C}}(g,f,1) = \min\left(\frac{1}{2}\left(1+\frac{1}{2}\right), \frac{1}{2}\left(1+\frac{1}{2p}\right)\right) = \frac{1}{2} + \frac{1}{4p}.$$

While all of the above cases corroborate Varchenko's result, the case $\lambda < 0$ deserves special attention. The above analysis provides the optimal δ even in this case, in contrast to [5], where it was only shown that

$$\delta < \frac{2p}{4p-2}.$$

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