Gromov's Random Groups have Property (T)

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Random Groups

Why random groups?

- Understanding discrete groups via presentations.
- Trying to get a feel for the "typical" group in some sense.
- Counterexamples.

Models for random groups: add relations "at random" to a given group.

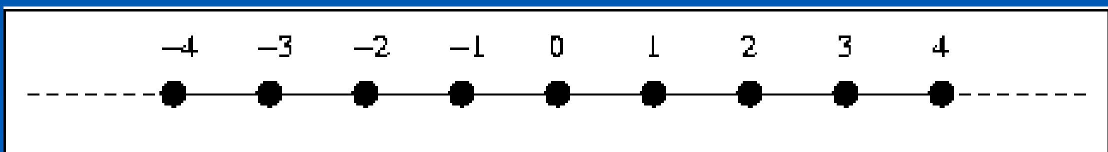
The Cayley graph $\mathbf{Cay}(\Gamma; S)$

Let Γ be a group, $S \subset \Gamma$ a finite subset. Decleare $(\gamma_1, \gamma_2) \in \Gamma^2$ to be an edge iff there exists $s \in S$ such that $\gamma_1 s = \gamma_2$. $\mathrm{Cay}(\Gamma; S)$ is:

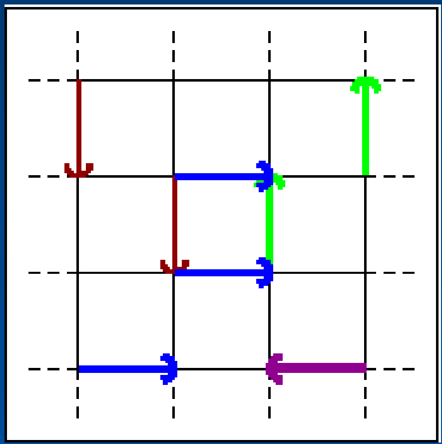
1. $|S|$-regular;
2. Connected iff S generates Γ;
3. Undirected iff S is symmetric (meaning $s \in S$ implies $s^{-1} \in S$).

From now on we assume S generates Γ, and is of the form $\{s_i^{\pm 1}\}_{i=1}^{k}$ (with possible repetitions).

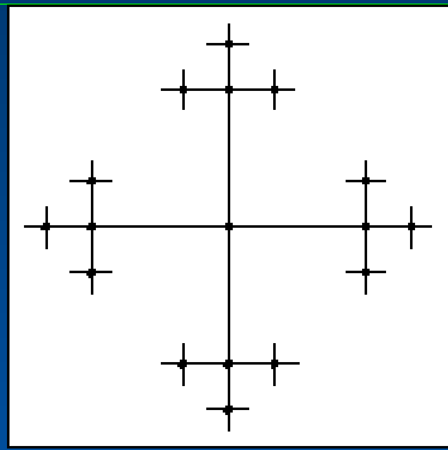
Example: $\Gamma = \mathbb{Z}$, $S = \{\pm 1\}$.



Examples:



$\Gamma = \mathbb{Z}^2$, $S = \{(\pm 1, 0),\ (0, \pm 1)\}$



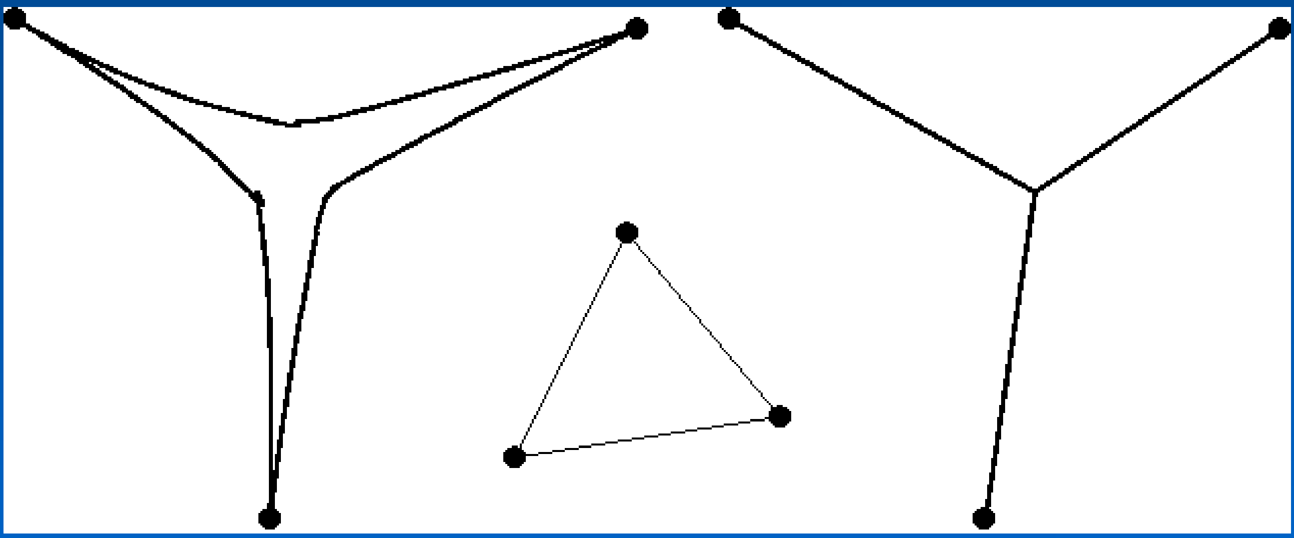
$\Gamma = F_{\{a,b\}}$, $S = \{a, a^{-1}, b, b^{-1}\}$

- $\mathrm{Cay}(\Gamma; S)$ has a symmetric “S-coloring” . This is Γ-invariant.
- Colored cycles of $\mathrm{Cay}(\Gamma; S)$ are *relations* (words in S which represents the trivial element).

Let R be a set of words in S. Call $\langle S \mid R \rangle$ a *presentation* of Γ if S generates Γ, R generate the group of relations.

Hyperbolicity (the geometry of Γ)

- "Coarse" geometry: call Γ *Gromov hyperbolic* if geodesic triangles in $\mathrm{Cay}(\Gamma; S)$ are δ-thin for some $\delta > 0$.
- Independent of the choice of S.
- **Example:** Free groups (trees): every geodesic triangle is 0-thin.



Property (T) (the geometry of actions of Γ)

- Let Γ act isometrically on (Y, d_Y). $y \in Y$ is ε*-almost invariant* if $d_Y(sy, y) \leq \varepsilon$ for all $s \in S$.

$$\iff \quad E(y) = \frac{1}{2|S|} \sum_{s \in S} \|sy - y\|_Y^2 \text{ is small.}$$

- *Kazhdan's property (T) with Kazhdan constant* ε*:* every unitary representation of with an ε-almost invariant unit vector has a Γ-invariant unit vector.
- **Example:** $\mathrm{SL}_n(\mathbb{Z})$ for $n \geq 3$, but not F_n.
- (Guichardet-Delorme) Γ is Kazhdan (T) iff every isometric action on a Hilbert space Y has a fixed point.
- (Margulis) Let Γ have property (T). Then $\{\mathrm{Cay}(\Gamma/N; S)\}_{N \lhd \Gamma}$ is family of expanders.

Models of random groups

Start with $\Gamma = \langle S \mid R \rangle$, and add relations.

In each case we have a sequence of probability spaces $\{(\mathcal{A}_m, \mathrm{Pr}_m)\}_{m=1}^{\infty}$ and for each $\alpha \in \mathcal{A}_m$ of words W_α in S, giving random groups:

$$\Gamma_\alpha = \langle S \mid R \cup W_\alpha \rangle .$$

As usual say $\mathcal{P}(\Gamma_\alpha)$ holds *asymptotically almost surely* (a.a.s) if

$$\lim_{m \to \infty} \mathrm{Pr}_m \{\alpha \in \mathcal{A}_m \mid \Gamma_\alpha \text{ has } \mathcal{P}\} = 1.$$

“Density” models

- Choose a parameter $0 \leq \delta \leq 1$
- Set $R_m =$ "all (reduced / cyclically reduced / geodesic) words of length m".
- Set $\mathcal{A}_m = \{\text{all subsets of } R_m \text{ of size } \approx |R_m|^\delta\}$.

Theorem (Ollivier) Let Γ be non-elementary hyperbolic. In each case there exists δ_c such that:

1. If $\delta > \delta_c$ then a.a.s. $|\Gamma_\alpha| \leq 2$.
2. If $\delta < \delta_c$ then a.a.s. Γ_α is non-elementary hyperbolic.

Proof: “Small cancellation theory” for hyperbolic groups.

- For $\Gamma = F_n$ the case of finitely many relations (“$\delta = 0$”) is due to Ol’shankii and Champetier, and the fact that $\delta_c = \frac{1}{2}$ is due to Gromov.

The “graph” model

- Take a graph $G = (V, E)$, let $\mathcal{A}_G = \left\{ \textit{symmetric}\ \alpha : \vec{E} \to S \right\}$.
- If $\vec{p} = (\vec{e}_1, \ldots, \vec{e}_r)$ is a path set $\alpha(\vec{p}) = \alpha(\vec{e}_1) \cdot \ldots \cdot \alpha(\vec{e}_r)$, and

$$W_\alpha = \{\alpha(\vec{c}) \mid \vec{c} \text{ a closed path in } G\}.$$

- **Example**: $\Gamma = F_{\{a,b\}}$, $G = C_4$ marked $aba^{-1}b^{-1}$. Then $\Gamma_\alpha \simeq \mathbb{Z}^2$.

Key observation: Let $X_\alpha = \mathrm{Cay}(\Gamma_\alpha; S)$. Take base vertices $u_0 \in V$, $x_0 \in X_\alpha$. For a path $\vec{p} : u_0 \rightsquigarrow u$ in G set

$$\alpha_{u_0 \to x_0}(u) = x_0 \alpha(\vec{p}).$$

This is a map of *decorated graphs* $\alpha_{u_0 \to x_0} : (G, \alpha) \to X_\alpha$.

Restricting to a neighbourhood of radius $< \frac{g}{2}$, we also have a well-defined map $B^G_{g/2}(u) \to X = \mathrm{Cay}(\Gamma; S)$.

Results

For simplicity assume G is d-regular, $d \geq 3$, and let $g = \mathrm{girth}(G)$.

Theorem 1: (Ollivier, Delzant) Assume that $g \gg \log|V|$ and that Γ is non-elementary hyperbolic. Then Γ_α is non-elementary hyperbolic a.a.s. as $|V| \to \infty$.

Also gives information about the radius of the injectivity of the quotient map $\Gamma \twoheadrightarrow \Gamma_\alpha$ and the maps $\alpha_{u_0 \to x_0}$.

Theorem 2: (S) Given $2k = |S|$, $0 \leq \lambda_0 < 1$ there exists g_0 such that if $g(G) \geq g_0$ and $\lambda^2(G) \leq \lambda_0^2$ then for some a, b depending on λ_0, k, d

$$\Pr\{\Gamma_\alpha \text{ has Kazhdan (T)}\} \geq 1 - ae^{-b|V|}.$$

Here $\lambda^2(G) = \max\{\lambda_i^2 \mid \lambda_i \neq \pm 1 \text{ is an eigenvalue of } G\}$.

Remarks:

1. $\Gamma_\alpha = \langle S \mid R \cup W_\alpha \rangle$ is a quotient of $\langle S \mid W_\alpha \rangle \Rightarrow$ w.l.g. Γ is free, so $X = \mathrm{Cay}(\Gamma; S)$ is a $2k$-regular tree.

2. Theorem 1 in fact uses a graph G' constructed from G by “blowing up” edges. Theorem 2 still holds in that context.

3. The construction seems to need expanders of large girth, i.e. the Ramanujan graphs of L-P-S.

4. Iterating the construction using larger and larger expanders we can form a limit group. With positive probability the Cayley graph of this “wild” group cannot be embedded in Hilbert space with bounded distortion.

Proof of Theorem 2

Averaging on G

- $\mu_G^n(u \to u')$ - Transition probability for n steps of the standard random walk on G. E.g.:

$$\mu_G(u \to u') = \mu_G^1(u \to u') = \begin{cases} \frac{1}{d} & (u,u') \in E \\ 0 & (u,u') \notin E \end{cases}.$$

- Let $f : V \to Y$ be a vector-valued function. Its "local averages" are defined by:

$$(A_{\mu_G^n} f)(u) = \sum_{u'} \mu_G^n(u \to u') f(u').$$

- Spectrum of G = Spectrum of A_{μ_G}. $\lambda^2(G)$ controls convergence of $A^n_{\mu_G^2} f$ to the constant function.

- “Proof”: Use energy (variance):

$$\begin{aligned} E_{\mu_G^n}(f) &= \frac{1}{2|V|}\sum_u \sum_{u'} \mu_G^n(u \to u')\, \|f(u) - f(u')\|_Y^2 \\ &= \left\langle \left(I - A_{\mu_G}^n\right) f, f \right\rangle \end{aligned}$$

- Expanding f in an eigenbasis we see:

$$E_{\mu_G^{2n}}(f) \le \frac{1}{1-\lambda^2(G)} E_{\mu_G^2}(f)$$

(independently of n!)

- Geometric fact: if $g \gg n$, then for $\mu_G^{4n}(u \to \cdot)$-most u',

$$\left|\mu_G^{4n}(u \to u') - \mu_G^{4n+2}(u \to u')\right| = o\left(\mu_G^{4n}(u \to u')\right)$$

- $\Rightarrow$ “Smoothing effect” of A_μ:

$$\begin{aligned} E_{\mu_G^2}(A_{\mu_G^{2n}} f) &= E_{\mu_G^{4n}}(f) - E_{\mu_G^{4n+2}}(f) \\ &= o_n(1) E_{\mu_G^{4n}}(f) + o_n(1) E_{\mu_G^2}(f) = o_n(1) E_{\mu_G^2}(f) \end{aligned}$$

Averaging on X

Representation of Γ_α on $Y \iff \Gamma$ acts on Y by isometries, W_α acting trivially. $\mu_X^{2l}(x \to x')$ std. rw. on X.

- To each $y \in Y$ associate $f_y(\gamma) = \gamma y$ (a Γ-equivariant map $f \colon X \to Y$).
- Note that $x \mapsto \sum_{x'} \mu_X^{2l}(x \to x') \|f(x) - f(x')\|_Y^2$ is Γ-invariant. Set

$$E_{\mu_X^{2l}}(f) = \frac{1}{2} \sum_{x \in \Gamma \backslash X} \sum_{x'} \mu_X^{2l}(x \to x') \|f(x) - f(x')\|_Y^2 .$$

  Then y almost-invariant $\iff$ $E_{\mu_X}(f)$ small.
- We need to find $y \in Y$, i.e. f, s.t. $E_{\mu_X}(f) = 0$. Idea: Fix $r < 1$. Given f, find l for which

$$E_{\mu_X^2}(A_{\mu_X^{2l}} f) \leq r E_{\mu_X^2}(f).$$

Proof of Theorem 2

Prop. 1: (Geometry) There exist $c_l, d_l \xrightarrow[l\to\infty]{} 0$ such that for every Γ-space Y and every equivariant $f\colon X \to Y$,

$$E_{\mu_X^2}(A_{\mu_X^{2l}} f) \leq c_l E_{\mu_X^{2l}}(f) + d_l E_{\mu_X^2}(f).$$

Prop. 2: (Spectral Gap) Assume $\frac{1}{2}g(G) > 2n$. Then a.a.s. as $|V| \to \infty$, for every Γ_α-space Y and every equivariant $f\colon X \to Y$, there exists $\frac{n}{6} \leq l \leq n$, such that:

$$E_{\mu_X^{2l}}(f) \leq \frac{10.5}{1-\lambda^2(G)} E_{\mu_X^2}(f).$$

Proof of Thm: Choose $n = 6l_0$ large enough such that $r = \frac{10.5}{1-\lambda_0^2} c_{l_0} + d_{l_0} < 1$, and let $g(G) > 4n$. Given Y and $f_j \in B^\Gamma(X,Y)$ we can set $f_{j+1} = A_{\mu_X^{2l}} f_j$ with the right l to get:

$$E_{\mu_X^2}(f_{j+1}) \leq r E_{\mu_X^2}(f).$$

Proposition 2

Recall the maps $\alpha_{u_0 \to x_0}\colon (G,\alpha) \to X_\alpha$. Since every $f : B^\Gamma(X,Y)$ decends to X_α we can consider $f \circ \alpha_{u_0 \to x_0}\colon G \to V$. By direct computation

$$E_{\mu_G^{2n}}(f \circ \alpha_{u_0 \to x_0}) = E_{\bar{\mu}_{X,\alpha}^{2n}}(f)$$

with the "effective walk"

$$\bar{\mu}_{X,\alpha}^{2n}(x \to x') = \frac{1}{|V|} \sum_{\substack{\alpha_{p_0 \to x}(\vec{p}) = x' \\ |\vec{p}| = 2n}} \mu_G^{2n}(\vec{p})$$

and the sum is over all paths $\vec{p}$ of length $2n$ connecting vertices u, u' such that $\alpha_{u \to x}(\vec{p}) = u'$.

- Since $\frac{1}{2}g(G) > 2n$, $\alpha_{u \to x}(\vec{p})$ is a well-defined member of X.

- Warning: $\bar{\mu}^{2n}_{X,\alpha} \neq \mu^{2n}_X$! In fact, $\bar{\mu}^{2n}_{X,\alpha} \approx \mu_X^{2n\cdot\frac{k-1}{k}}$.

For any $x, x' \in X$ we now consider $\bar{\mu}^{2n}_{X,\alpha}$ as a function of α. From the radial symmetry it is clear that we can write:

$$\mathbb{E}\bar{\mu}^{2n}_{X,\alpha}(x \to x') = \sum_l p_l \mu^{2l}_X(x \to x'),$$

- If $\frac{1}{2}g(G) > 2n$ we can take $l \geq \frac{n}{6}$ without much error.
- Observation: changing a single $\alpha(e)$ makes a very small change to $\bar{\mu}^{2n}_{X,\alpha}$. Also, $\mathcal{A}_G$ is the product measure space.

- Hence (concentration of measure and the union bound) all the $\bar{\mu}^{2n}_{X,\alpha}(x \to x')$ are close to their expectation values with high probability. If that happens then for every Y and f as above,

$$\frac{1}{2}\sum_{\frac{n}{6}\le l} p_l E_{\mu^{2l}_X}(f) \le \frac{1}{2}\mathbb{E}\bar{\mu}^{2n}_{X,\alpha}(x \to x') \le E_{\bar{\mu}^{2n}_{X,\alpha}}(f).$$

By lifting f to G we know

$$E_{\bar{\mu}^{2n}_{X,\alpha}}(f) \le \frac{1}{1-\lambda^2(G)} E_{\bar{\mu}^{2}_{X,\alpha}}(f),$$

and under the "high probability" umberlla we can also add $E_{\bar{\mu}^{2}_{X,\alpha}}(f) \le E_{\mu^{2}_X}(f)$, concluding that

$$\frac{\sum_{\frac{n}{6}\le l} p_l E_{\mu^{2l}_X}(f)}{\sum_{\frac{n}{6}\le l} p_l} \le \frac{2}{\sum_{\frac{n}{6}\le l} p_l} \cdot \frac{1}{1-\lambda^2(G)} E_{\mu^{2}_X}(f),$$

Finally, the smallest $E_{\mu^{2l}_X}(f)$ must be at least their average.

Proposition 1

μ random walk on X, Y a $\mathrm{CAT}(0)$ space, $f : X \to Y$ Γ-equivariant. ν a Γ-invariant measure on X. H_n denotes A_{μ^n} here.

$$E_\mu(H_n f) \leq \frac{1}{2} \int_{\bar{X}} d\bar{\nu}(x) \int_X \left[d\mu^{n+1}(x \to x') - d\mu^n(x \to x') \right] d_Y^2(H_n f(x), f(x')),$$

and

$$\frac{1}{2} \int_{\bar{X}} d\bar{\nu}(x) \int_X d\mu^n(x \to x') d_Y^2(H_n f(x), f(x')) \leq E_{\mu^n}(f).$$

Use with $\mu = \mu_X^2$. Since X is a tree, $\mu_X^{2n+2}(x \to x') - \mu_X^{2n}(x \to x')$ is small.