THE FUNDAMENTAL INTERCONNECTENESS OF ALL THINGS (MATHEMATICAL)

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Notes for undergraduate colloquium talk.

INTRODUCTION

Idea: we only have one mathematics. Sometimes not clear from courses.
As a motiativing them we'll consider one problem. Ready bingo cards for math fields.

Problem 1. Which integers can be represented as a sum of four squares?

Define $r_k(n) = \#\left\{\underline{x} \in \mathbb{Z}^k \mid \sum_{i=1}^k x_i^2 = n\right\}$.

(1) $k = 1$ these are the squares
(2) $k = 2$ Fermat; unique factorization in $\mathbb{Z}[i]$.
(3) $k = 3$ Lagrange: $r_k(n) = 1$ iff $n < 0$ or $n = 4^a(8m+7)$.
(4) $k = 4$ Legendre: $r_k(n) \geq 1$ iff $n \geq 0$ (Euler identity / factorization of quaternions + proof for primes).

1. COMBINATORICS

Observe that $r_{k+\ell}(n) = \sum_{a+b=n} r_k(a) r_\ell(b)$ – *additive convolution*. We therefore consider the *generating function*

$$\theta_k(q) = \sum_{n=0}^{\infty} r_k(n) q^n \in \mathbb{Z}[[q]]\,.$$

We have

$$\begin{aligned}\sum_{n\geq 0} r_{k+\ell}(n) q^n &= \sum_{n\geq 0} q^n \sum_{a+b=n} r_k(a) r_\ell(b) \\ &= \sum_{n\geq 0} \sum_{a+b=n} r_k(a) q^a r_\ell(b) q^b \\ &= \sum_a r_k(a) q^a \sum_b r_\ell(b) q^b\,.\end{aligned}$$

Thus $\theta_k(q) = (\theta(q))^k$ for $\theta = \theta_1$.

2. ANALYSIS

2.1. **Complex analysis 1.** The series converges for $|q| < 1$, but no good analytic properties from that point of view. Instead try $q = e(\tau) = e^{2\pi i \tau}$ so

$$\theta(\tau) = \sum_{d\in\mathbb{Z}} e^{2\pi i d^2 \tau}\,.$$

We have $|q| < 1$ iff $\Im\tau > 0$ so this is a function on the *upper halfplane*. The series converges locally uniformly absolutely there, so defines a holomorphic function.

2.2. **Harmonic analysis.** Let $\tau = iy$. Let $f(x) = e^{-2\pi y \cdot x^2}$ and let $F(x) = \sum_{d\in\mathbb{Z}} f(x+d)$ (so that $\theta(iy) = F(0)$). Then $F(x)$ is a smooth function on the circle $\mathbb{R}/\mathbb{Z}$ so Fourier analysis gives

$$\begin{aligned}F(x) &= \sum_{k\in\mathbb{Z}} \hat{F}(k) e(kx) \\ &= \sum_{k\in\mathbb{Z}} \hat{f}(k) e(kx)\,.\end{aligned}$$

Setting $x = 0$ we get the *Poisson summation formula*:

$$\sum_{d\in\mathbb{Z}} f(d) = \sum_{k\in\mathbb{Z}} \hat{f}(k)\,.$$

Here

$$\begin{aligned}\hat{f}(k) &= \int_{-\infty}^{+\infty} e^{-2\pi yx^2} e^{-2\pi ikx} dx \\ &= e^{-2\pi k^2/4y} \int_{-\infty}^{+\infty} e^{-2\pi y(x^2-\frac{ik}{y}x-\frac{k^2}{4y^2})} dx \\ &= \frac{1}{\sqrt{-2i\tau}} e^{2\pi ik^2/(-4\tau)}\end{aligned}$$

so for $\tau = iy$ we get

$$\theta\left(-\frac{1}{4\tau}\right) = \sqrt{-2i\tau}\theta(\tau)\,.$$

2.3. **Digression (Riemann).** Consider $\frac{1}{2}\int_0^\infty (\theta(iy)-1)\, y^{2s}\frac{dy}{y}$. For $\Re(s) > 1$ we have

$$\begin{aligned}\int_0^\infty \sum_{d=1}^{\infty} e^{-2\pi d^2 y} y^{2s}\frac{dy}{y} &= \sum_{d=1}^{\infty}\int_0^\infty e^{-2\pi d^2 y} y^{2s}\frac{dy}{y} \\ &= \sum_{d=1}^{\infty} d^{-s}\int_0^\infty e^{-2\pi y} y^{2s}\frac{dy}{y} \\ &= \zeta(s)\zeta_\infty(s)\end{aligned}$$

and now the transformation rule can be used to prove the analyical contiuation and functional equation for $\zeta(s)$.

2.4. **Complex analysis 2.** By analyticity we have

$$\theta\left(-\frac{1}{4\tau}\right) = \sqrt{-2i\tau}\theta(\tau)$$

for all τ. Using invariance by $\tau \to \tau + 1$ we also get[1]

$$\theta\left(\frac{\tau}{4\tau+1}\right) = \sqrt{4\tau+1}\theta(\tau)\,.$$

2.5. **Group theory and geometry.** $\mathrm{SL}_2\mathbb{R}$ acts on $\mathbb{H}$ via $\gamma\tau = \frac{a\tau+b}{c\tau+d}$. Thus let $\Gamma_\theta < \mathrm{SL}_2\mathbb{Z}$ be the subgroup generated by $\tau \to \tau+1$ and $\tau \to \frac{\tau}{4\tau+1}$. We get

$$\theta(\gamma z) = \sqrt{c\tau+d}\,\theta(\tau)$$

for all $\gamma \in \Gamma_\theta$.

Fact 2. $\Gamma_\theta = \Gamma_0(4) = \left\{\begin{pmatrix} a & b \\ c & d\end{pmatrix} \equiv \begin{pmatrix} 1 & * \\ 0 & 1\end{pmatrix}(4)\right\}$.

NOT FOR TALK. Clearly $\Gamma_\theta \subset \Gamma_0(4)$. Now let $\gamma = \begin{pmatrix} a & b \\ c & d\end{pmatrix} \in \Gamma_0(4)\setminus\Gamma_\theta$ have $|c|$ minimal, and subject to have that $|d|$ minimal. Then $c \neq 0$ (otherwise $\gamma = \pm\begin{pmatrix} 1 & b \\ 0 & 1\end{pmatrix} \in \Gamma_\theta$). Then $\begin{pmatrix} a & b \\ c & d\end{pmatrix}\begin{pmatrix} 1 & n \\ 0 & 1\end{pmatrix} = \begin{pmatrix} * & * \\ c & d+nc\end{pmatrix} \in \Gamma_0(4)\setminus\Gamma_\theta$ and from the minimality of $|d|$ we get that $|d| \leq \frac{1}{2}|c|$ and the inequality is strict since $4 \mid c$ and $d \equiv 1\,(4)$ is odd. Similarly $\begin{pmatrix} a & b \\ c & d\end{pmatrix}\begin{pmatrix} 1 & 0 \\ 4n & 1\end{pmatrix} = \begin{pmatrix} * & * \\ c+4nd & d\end{pmatrix} \in \Gamma_0(4)\setminus\Gamma_\theta$ and from the minimality of c we get $|c| \leq 2|d| < |c|$, a contradiction. □

[1] using $\begin{pmatrix} 0 & 1/4 \\ -1 & 0\end{pmatrix}\begin{pmatrix} 1 & -1 \\ 0 & 1\end{pmatrix}\begin{pmatrix} 0 & -1 \\ 4 & 0\end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 4 & 1\end{pmatrix}$

2.6. **Complex analysis 3.** $\theta_4 = \theta^4$ satisfies

$$\theta_4(\gamma z) = (c\tau + d)^2\,\theta(\tau)\,. \tag{2.1}$$

There are not functions on $\Gamma_\theta\backslash\mathbb{H}$! But they are sections of a *line bundle*.

Fact 3. *(Riemann–Roch) The space of functions satisfying* (2.1) *and holomorphic at infinity is* 2*-dimensional.*

3. THE SPACE OF LATTICES

$$G_2(\Lambda) = \sum_{\lambda\in\Lambda\backslash\{0\}} \lambda^{-2}$$

is not absolutely convergent. Works conditionally if we order

$$G_2(\Lambda_\tau) = \sum_{c\in\mathbb{Z}} \sum_{d\in\mathbb{Z}}{}' \frac{1}{(c\tau+d)^2}$$

Then (with σ the divisor function)

$$G_2(\tau) = 2\zeta(2) - 8\pi^2 \sum_{n=1}^{\infty} \sigma(n) q^n\,.$$

Now G_2 is not invariant!

$$\frac{1}{(c\tau+d)^2} G_2(\gamma\tau) = G_2(\tau) - \frac{2\pi i c}{c\tau+d}$$

(conditional convergence FTW). But this means

$$G_{2,2}(\tau) = G_2(\tau) - 2G_2(2\tau)$$
$$G_{2,4}(\tau) = G_2(\tau) - 4G_2(4\tau)$$

are $\Gamma_0(4)$-invariant. Compute Fourier expansion, use first few coefficients to see $\theta_4 = -\frac{1}{\pi^2} G_{2,4}$ and get

$$r_4(n) = 8 \sum_{\substack{0<d\mid n\\ 4\nmid d}} d\,.$$

- Can solve 2-square problem, 6-square problem, up to 10 square problem the same way.
- 12-square problem is different.

3.1. **Algebraic geometry.** $\Gamma_\theta\backslash\mathbb{H}$ can be given the structure of an algebraic variety.

3.2. **Representation theory, PDE.** Space of lattices => representation of $\mathrm{SL}_2(\mathbb{R})$
CR equation => representation is *irreducible*.