Math 100:V02 – WORKSHEET 12
DIFFERENTIAL EQUATIONS

1. Differential equations

(1) For each equation: Is $y = 3$ a solution? Is $y = 2$ a solution? What are *all* the solutions?

$$y^2 = 4 \qquad ; \qquad y^2 = 3y$$

(2) For each equation: Is $y(x) = x^2$ a solution? Is $y(x) = e^x$ a solution?

$$\frac{dy}{dx} = y \qquad ; \qquad \left(\frac{dy}{dx}\right)^2 = 4y$$

(3) Which of the following (if any) is a solution of $\frac{dz}{dt} + t^2 - 1 = z$ (challenge: find more solutions):

A. $z(t) = t^2$; B. $z(t) = t^2 + 2t + 1$

(4) Which of the following (if any) is a solution of $\frac{dy}{dx} = \frac{x}{y}$

A. $y = -x$; B. $y = x + 5$ C. $y = \sqrt{x^2 + 5}$

(5) The balance of a bank account satisfies the differential equation $\frac{dy}{dt} = 1.04y$ (this represents interest of 4% compounded continuously). Sketch the solutions to the differential equation. What is the solution for which $y(0) = \$100$?

(6) Suppose $\frac{dy}{dx} = ay$, $\frac{dz}{dx} = bz$. Can you find a differential equation satisfied by $w = \frac{y}{z}$? Hint: calculate $\frac{dw}{dx}$.

Date: 29/2/2024, Worksheet by Lior Silberman. This instructional material is excluded from the terms of UBC Policy 81.

2. Solutions by massaging and ansatze

(7) For which value of the constant ω is $y(t) = \sin(\omega t)$ a solution of the oscillation equation $\frac{d^2y}{dt^2} + 4y = 0$?

(8) (The quantum harmonic oscillator) For which value of the constants A, B (with $B > 0$) does the function $f(x) = Axe^{-Bx^2}$ satisfy $-f'' + x^2 f = 3f$? What if we also insist that $f(1) = 1$?

(9) Consider the equation $\frac{dy}{dt} = a(y - b)$.

(a) Define a new function $u(t) = y(t) - b$. What is the differential equation satisfied by u?

(b) What is the general solution for $u(t)$?

(c) What is the general solution for $y(t)$?

(d) Suppose $a < 0$. What is the asymptotic behaviour of the solution as $t \to \infty$?

(e) Suppose we are given the *initial value* $y(0)$. What is C? What is the formula for $y(t)$ using this?

(10) Example: *Newton's law of cooling.* Suppose we place an object of temperature $T(0)$ in an environment of temperature T_{env}. It turns out that a good model for the temperature $T(t)$ of the object at time t is

$$\frac{dT}{dt} = -k\left(T - T_{\text{env}}\right)$$

where $k > 0$ is a positive constant.

(a) Suppose $T(t) > T_{\text{env}}$. Is $T'(t)$ positive or negative? What if $T(t) < T_{\text{env}}$? Explain this in words.

(b) A body is found at 1:30am and its temperature is measured to be 32.5°C. At 2:30am its temperature is found to be 30.3°C. The temperature of the room in which the body was found is measured to be 20°C and we have no reason to believe the ambient temperature has changed. What was the time of death?

(11) A body falling through the air is at height $y(t)$ at time t where $y(t)$ satisfies the differential equation

$$\frac{d^2y}{dt^2} = -g + \kappa\left(\frac{dy}{dt}\right)^2 .$$

Here g is the acceleration due to gravity and κ is the *drag coefficient.*

(a) Write the differential equation satisfied by the velocity $v = \frac{dy}{dt}$.

(b) This differential equation has a *fixed point* (also known as a *steady state*): find the value u (called the "terminal velocity") such that the constant function $v(t) \equiv u$ is a solution.

(c) Define the *hyperbolic trigonometric* functions $\cosh x = \frac{e^x+e^{-x}}{2}$, $\sinh x = \frac{e^x-e^{-x}}{2}$, and $\tanh x = \frac{\sinh x}{\cosh x}$. Check that $(\cosh x)' = \sinh x$, $(\sinh x)' = \cosh x$ and that $(\tanh x)' = 1 - \tanh^2 x$.

(d) Find the values of A, α for which

$$v = -A\tanh\left(\alpha(t - t_0)\right)$$

solves the differential equation.

(e) Show that $\lim_{x\to\infty} \tanh x = 1$ and conclude that $v(t)$ indeed converges to the terminal velocity as $t \to \infty$.